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D-brane probes of special holonomy manifolds, and dynamics of $\mathcal{N} = 1$ three-dimensional gauge theories

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Abstract: Using D2-brane probes, we study various properties of M-theory on singular, non-compact manifolds of G_2 and Spin(7) holonomy. We derive mirror pairs of $\mathcal{N} = 1$ supersymmetric three-dimensional gauge theories, and apply this technique to realize exceptional holonomy manifolds as both Coulomb and Higgs branches of the D2-brane world-volume theory. We derive a "G_2 quotient construction" of non-compact manifolds which admit a metric of G_2 holonomy. We further discuss the moduli space of such manifolds, including the structure of geometrical transitions in each case. For completeness, we also include familiar examples of manifolds with SU(3) and Sp(2) holonomy, where some of the new ideas are clarified and tested.

Keywords: Field Theories in Lower Dimensions, Duality in Gauge Field Theories, Brane Dynamics in Gauge Theories.

Contents

1. Introduction and summary

D-brane probes of non-compact Calabi-Yau manifolds have proven to be powerful tools in understanding the dynamics of both string theory and supersymmetric gauge theories [1]. The purpose of this paper is to extend some of these results to cases with less supersymmetry, where the background manifold has exceptional holonomy.

When M-theory is compactified on a smooth manifold X of exceptional holonomy, with all typical scales much larger than the Planck scale, the supergravity approximation is valid and one can simply derive the low-energy effective theory by the familiar rules of the Kaluza-Klein reduction [2]. This leads to a rather simple effective theory containing, in particular, only abelian gauge fields. To get more interesting physical phenomena, such as non-abelian gauge symmetry and phase transitions, we must take a limit where the manifold X develops a singularity. Since the physics associated with singularity is local, i.e. does not depend very much on the details of the smooth part of the manifold, one can study such phenomena by isolating the singular region of X and studying M-theory or string theory on a non-compact model of X. For this reason, it is interesting to understand M-theory dynamics on non-compact manifolds of special holonomy.

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$Hol(X)$	$\dim(X)$	SUSY on $D2$-brane	Examples of X
SU(3)	6	$\mathcal{N}=2$	The conifold
G_2	7	$\mathcal{N}=1$	The cone over $\mathbb{CP}^3$
			The cone over $\mathrm{SU}(3)/\mathrm{U}(1)^2$
			New manifold with $h_2+h_3=2$
			New manifold with $h_2+h_3=3$
			The cone over $\mathbf{S}^3\times\mathbf{S}^3$
Sp(2)	8	$\mathcal{N}=3$	$T^*\mathbb{CP}^2$, $T^*\mathcal{B}_n$
Spin(7)	8	$\mathcal{N}=1$	New manifolds with $h_2+h_3\geq 3$

Table 1: A list of models analyzed in this paper.

In this paper we will consider M-theory on a non-compact manifold X of holonomy SU(3), G_2, Sp(2) and Spin(7), such that X has real dimension $\dim(X) = 6, 7, 8$ and 8 respectively (see table 1). To determine the physics of M-theory on X, it is often useful to reduce to IIA on a spatial $\mathbf{S}^1$. There are essentially two possibilities: we may either choose the $\mathbf{S}^1$ to be transverse to X, or to be embedded within X. These are depicted in figure 1, where, starting at the top of the figure, we move clockwise or anti-clockwise respectively. In the former case, we simply end up with IIA string theory compactified on X. In the latter case however, the resulting IIA background, $X/\mathrm{U}(1)$, depends strongly on the choice of the circle action. A particularly convenient choice of embedding $\mathbf{S}^1$ — which we call L-picture, following [3, 4] — occurs when the resulting IIA space-time geometry is topologically flat:

$$X/\mathrm{U}(1) \cong \mathbb{R}^n\,, \qquad n = \dim(X) - 1\,. \tag{1.1}$$

If such a quotient exists, all the information about the topology of X is encoded in the fixed point locus, which we denote L (thus giving the name to the L-picture). From the IIA perspective, these fixed points are the positions of D6-branes lying in the directions transverse to $X/\mathrm{U}(1)$:

$$M^{10-n}\times L \subset M^{10-n}\times\mathbb{R}^n\,, \tag{1.2}$$

where M is Minkowski space. This ensures that the locus L has dimension

$$\dim(L) = \dim(X) - 4\,. \tag{1.3}$$

The task of identifying the geometry of L directly is rather hard. It has been undertaken recently for some examples of G_2 [3] and Spin(7) [4] manifolds. However, as explained in [4], under the assumption that such an L-picture exists, the homology of the fixed point set can be easily determined from the homology of X by means of the following general formulas,

$$\begin{aligned} h_0(L) &= h_2(X)+1 \\ H_i(L;\mathbb{Z}) &\cong H_{i+2}(X;\mathbb{Z})\,, \qquad i>0 \end{aligned} \tag{1.4}$$

These formulas, which were derived by matching the BPS states in the IIA and M-theory picture, allow us to simply write down the topology of the fixed point set L in many cases

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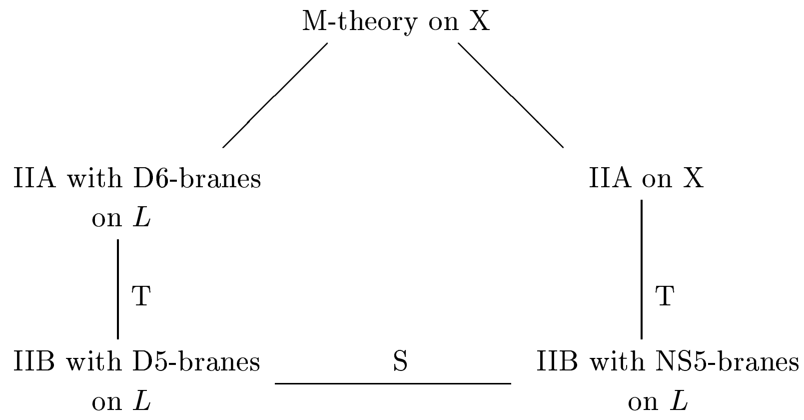

Figure 1: From M-theory to type-II strings and back: the web of dualities.

of interest. In the following we shall develop the L-picture in more detail, determining the explicit curve in several cases. We show that much of the physics of M-theory on X can be understood from this picture.

As shown in figure 1, we may connect the L-picture with the manifold X in two ways: either by returning to M-theory, or by performing a T-duality to IIB string theory. In the latter route, L is interpreted as a locus of NS5-branes which describes the T-dual of X. Familiar examples include the equivalence between parallel NS5-branes and A-type ALE spaces, and between orthogonal NS5-branes and (generalized) Calabi-Yau conifolds [5, 6]. Here we discuss in detail several G_2 examples.

One of the main themes of this paper is the story of brane probes on manifolds of exceptional holonomy. For M2-brane probes placed transverse to X of holonomy SU(3), G_2, $Sp(2)$ or Spin(7), the world-volume theory enjoys $\mathcal{N} = 2, 1, 3$ or 1 supersymmetry respectively in $d = (2+1)$ dimensions. If we reduce on a circle transverse to the M2-brane, we have a D2-brane probe of a IIA background. For the SU(3) and G_2 cases, we have two ways of performing this reduction, resulting in a D2-brane probe of the L picture, or a D2-brane probe of the manifold X. The two resulting $d = 2 + 1$ dimensional theories on the probe world-volumes are related by mirror symmetry, a duality of three-dimensional gauge theories which, among other things, exchanges Coulomb and Higgs branches [7]. In situations where we can successfully identify both world-volume theories, this provides a stringy strategy to derive large classes of three dimensional mirror pairs. In fact, it actually provides two such strategies, depending on the route we take around the circle in figure 1. We may either lift to M-theory, and return immediately to IIA on a different circle [8], a procedure which, in view of politically correct sensibilities, we refer to as the "M-theory flip". Alternatively, we may choose the IIB route, resulting in Hanany-Witten brane configurations [9]. This latter procedure has been used to great effect to derive mirror symmetry for various three dimensional gauge theories [10]–[16].

In the present paper, we employ a logic that is somewhat reversed from the description above; we use mirror symmetry of the field theory to derive aspects of the probe world-volume theory. We choose this method because, for manifolds of G_2 holonomy, the $\mathcal{N} = 1$ (two supercharges) theory on the world-volume of the D2-brane probe is not yet well understood. For G_2 orbifolds, and their partial resolutions, one may determine the probe theory using the techniques of [1] (see [17] for some initial work in this direction). However, for the conical singularities of (phenomenological) interest, this procedure does not work. A new technique is required, and the L picture provides this. In contrast to the probe of X, the world-volume theory of a D2-brane probe of the D6-brane locus L is, in many cases, extremely simple. In particular, in the limit in which X develops a conical singularity, the D6-brane locus L also degenerates into a cone [3], and often becomes a collection of flat, intersecting, D6 branes [18]. Each of these contributes a hypermultiplet to the D2-brane world-volume theory, the precise coupling of which breaks supersymmetry to $\mathcal{N} = 1$. The world-volume field theory operators corresponding to deformations away from the singular limit can then be identified, and the full quantum corrected Coulomb branch is conjectured to reproduce the manifold X, with the dual photon playing the role of the M-theory circle. At this point, we can invoke mirror symmetry of the gauge theory — derived using independent techniques — to write down a putative theory for the D2-brane probe of X. As we shall see, in many cases of interest, we can reconstruct the manifold X as the Higgs branch of this dual gauge theory.

Before proceeding to the detailed outline of the paper, it is worth making a few cautionary points. In particular, while the techniques of brane probes are tried and tested for theories with $\mathcal{N} \geq 2$ supersymmetry in three dimensions, one must necessarily be more skeptical when dealing with probes of exceptional holonomy manifolds. With only $\mathcal{N} = 1$ supersymmetry (2 supercharges), the power of holomorphy is lost, and with it our cherished non-renormalization theorems. Our hopes rest on the discrete parity symmetry,

$$P : (x^0, x^1, x^2) \to (x^0, -x^1, x^2) \tag{1.5}$$

under which both gauge field mass terms, as well as real superpotentials [19, 16, 20] are odd. Since the question of parity invariance of the theory is determined at one-loop [21, 22, 23], we may use this to prohibit the lifting of moduli spaces of vacua in some of our models. However, no further information is available. In particular, we know of no field theoretic reason that even the topology of the vacuum moduli space need agree with the space-time background although, at least for weakly curved backgrounds, we would expect this on physical grounds. Nevertheless, we shall see that using the techniques of mirror symmetry, we are able to reconstruct both the topology, and the isometries of the manifold X. This latter statement is particularly non-trivial since the quotient from M-theory to the IIA L-picture partially destroys the isometries, which are expected to be recovered only in the strong coupling limit. The fact that mirror symmetry of $\mathcal{N} = 1$ theories yields the full isometry group of X is, we believe, vindication of our methods. Finally, the most ambitious hope would be to recover the metric on X using these techniques. In particular, when applied to $d = 1+1$ dimensional theories, the G_2 quotient construction of section 4.1 (described in more detail below) provides a linear sigma model whose target space admits a metric of G_2 holonomy. In the infra-red, the theory necessarily flows to a (near) Ricci-flat metric. However, in the absence of something akin to Yau's theorem, we cannot be sure that this is indeed the metric of G_2 holonomy.

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The plan of the paper is as follows. In section 2, we review what is known about abelian mirror symmetry, and derive several classes of putative $\mathcal{N}=1$ mirror pairs, using both field theoretic as well as string theory techniques. We give several examples that will have useful applications in the sequel. Readers interested only in the brane probe theories, and not the methods used to derive them, may safely skip this section. We have also tried to make each subsequent section self-contained. For each manifold X listed in table 1, we follow a simple pattern. Firstly we identify the locus L of D6-branes, and write down the theory on a probe D2-brane, whose quantum corrected Coulomb branch realizes X. Secondly, we determine the mirror three-dimensional gauge theory and, thus, reconstruct X algebraically as a Higgs branch. In this fashion we work our way around the circle of figure 1.

Section 3 acts as a warm up exercise, where we review our methods as applied to the simplest $\mathcal{N}=2$ example which arises on a D2-brane probe of a Calabi-Yau conifold. We describe various aspects of deformations and mirror pairs in this case.

Section 4, describing G_2 manifolds, contains the main part of the paper. We start by reviewing the restrictions on flat special lagrangian planes which, physically, corresponds to the requirement that a collection of D6-branes preserves four supercharges. Upon lifting to M-theory, this results in a manifold X of G_2-holonomy. Using the techniques developed in section 2, we write down a linear sigma model whose target space is topologically X, and therefore admits a metric of G_2 holonomy. We refer to this as the "G_2-quotient construction".

The remainder of section 4 examines various examples of the G_2 quotient construction, starting with the cone on $\mathbb{CP}^3$ and the cone on $\mathrm{SU}(3)/\mathrm{U}(1)^2$, both of which were discussed by Atiyah and Witten [3]. We recover some of the results of Acharya and Witten [24] from a brane probe perspective. Moreover, we show that the cone over $\mathrm{SU}(3)/\mathrm{U}(1)^2$ has an extra, non-normalizable, moduli not considered in [3]. We then turn attention to two further examples where the L picture consists of three and four orthogonally intersecting D6-branes. We determine the homology of the M-theory lift X and, using mirror symmetry, derive algebraic descriptions of these manifolds as quotient spaces. These manifolds have a rich moduli space of (non-normalizable) deformations in which two cycles undergo flop transitions, or are replaced by three cycles.[1] From the L-picture it is clear that each such transition is inherited from the Calabi-Yau conifold discussed in section 3. Finally, we turn to the more subtle case where X is a cone over $\mathbf{S}^3 \times \mathbf{S}^3$. The theory on a D2-brane probe of this model has already been discussed by Aganagic and Vafa [34]. We elaborate on their construction and provide the mirror gauge theory.

In section 5 we discuss D2-brane probes of hyperKähler 8-manifolds with Sp(2) holonomy. This is partly to elucidate some of the issues unique to 8-dimensional spaces in preparation for the Spin(7) examples. We also clarify some outstanding issues about the probe theory and extend the results of [39] to hyperKähler singularities of the form $T^*\mathcal{B}_n$, where $\mathcal{B}_n$ is a del Pezzo surface. In particular, we find that for every $\mathcal{B}_n$ there is a model (with a special value of the G-flux), which has two vacua.

[1]For a recent discussion of geometric transitions in M-theory on G_2 holonomy manifolds see, for example [25]–[38].

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In the final section, we discuss manifolds of Spin(7) holonomy. We restrict ourselves to manifolds X whose L-picture consists of up to seven, mutually orthogonal D6-branes. This does not include any of the examples discussed in [4], and the explicit metric on these Spin(7) holonomy manifolds is not known. Nevertheless, these manifolds possess an intricate moduli space, including branches of different topologies in which two-cycles are exchanged for three-cycles. To our knowledge, this is the first example of a geometric transition in a Spin(7) manifold, albeit one which is locally equivalent to the Calabi-Yau conifold transition. Unlike for the above discussion of G_2 manifolds, there are now insufficient dimensions to perform an M-theory flip and string theory provides no method of deriving mirror pairs of three-dimensional gauge theories probing Spin(7) backgrounds. Nevertheless, our field theoretic techniques allow us to derive a mirror theory, providing an algebraic description of the manifolds.

Even though we mainly consider a single brane probe, we expect that some of our results can be generalized to non-abelian gauge theories on multiple branes. In particular, it would be interesting to extend mirror symmetry to such models, and explore various phases. We will not pursue this here. Let us just briefly mention that in the last two cases of Sp(2) and Spin(7) holonomy it is easy to predict what happens if we place a large number of membranes at the conical singularity of X. Namely, following the usual ideas of AdS/CFT correspondence [40], it is natural to expect three-dimensional conformal field theories with $\mathcal{N}=3$ and $\mathcal{N}=1$ supersymmetry, respectively. In fact, in the case of hyperKähler singularities there are more reasons to expect that the same happens at finite N [39].

We also include an appendix, containing a new class of mirror pairs for $\mathcal{N}=1$ Maxwell-Chern-Simons theories which, in particular, allow for the possibility of compact Coulomb branches. We illustrate this phenomenon with a theory which has an $\mathbf{S}^3$ Coulomb branch.

The results of sections 2.1 and 4.1, together with the example of section 4.3, are summarized in the companion paper [41].

2. Mirror symmetry in three dimensional gauge theories

Mirror symmetry of three dimensional gauge theories refers to a conjectured quantum equivalence between a pair of theories which, for the remainder of this paper, we shall refer to as Theory A and Theory B. The Coulomb branch of Theory A coincides with the Higgs branch of Theory B and vice versa. We start here with a review of the most general abelian $\mathcal{N}=4$ mirror pairs [7, 10, 42], and discuss the techniques of [42]–[45] which allow one to deform these theories to mirror pairs with less supersymmetry. The most general abelian mirror pair may be most simply derived using the Kapustin-Strassler formula [42], yielding

$$\begin{aligned}\textbf{Theory A}: &\ \ \mathrm{U}(1)^r \text{ with } N \text{ hypermultiplets}\\ \textbf{Theory B}: &\ \ \hat{\mathrm{U}(1)}^{N-r} \text{ with } N \text{ hypermultiplets}\end{aligned}$$

where the $\mathcal{N}=4$ vector multiplets contain a gauge field and a triplet of real scalars ϕ, together with four Majorana spinors. The $\mathcal{N}=4$ hypermultiplets also contain four

Majorana spinors, this time paired with a doublet of complex scalars w,

$$w = \begin{pmatrix} q \\ \tilde{q}^\dagger \end{pmatrix} \tag{2.1}$$

where q and $\tilde{q}$ are in conjugate representations of the gauge group. For Theory A we denote the charge of the hypermultiplets as R_i^a, while for Theory B it is $\hat{R}_i^p$, $i = 1, \ldots, N$; $a = 1, \ldots, r$; $p = 1, \ldots, N - r$. Each of these matrices is assumed to be of maximal rank. Mirror symmetry requires,

$$\sum_{i=1}^{N} R_i^a \hat{R}_i^p = 0 \qquad \forall\, a, p \tag{2.2}$$

We denote the coupling constant of the two theories as e^2 and $\hat{e}^2$ respectively. For simplicity we shall concentrate on the case where Theory A lies on its Coulomb branch and Theory B on its Higgs branch, each a $4r$ real dimensional hyperKähler manifold. The theories include further parameters consistent with supersymmetry: a triplet of masses, m_i, for each hypermultiplet of Theory A, and a triplet of FI parameters, ζ^p, for each gauge factor of Theory B. (Note that including FI parameters for Theory A or mass parameters for Theory B would partially lift the vacuum moduli space of interest, and so we set these to zero.) Note further that not all the mass parameters of Theory A are independent; precisely k of them may be absorbed by suitable shifts of the vector multiplet scalars.

The mirror map between operators is given by,

$$R_i^a \phi_a + m_i = \hat{w}_i^\dagger \tau \hat{w}_i\,, \qquad \hat{R}_i^p \hat{\phi}_p + \hat{m}_i = w_i^\dagger \tau w_i\,, \tag{2.3}$$

where τ is the triplet of Pauli matrices. The FI and mass parameters obey the similar relationship,

$$\zeta^p = \sum_i \hat{R}_i^p m_i\,. \tag{2.4}$$

With $\mathcal{N} = 4$ supersymmetry, non-renormalization theorems guarantee that the metric on the Higgs branch is classical, while that on the Coulomb branch receives no corrections beyond one-loop.[2] Each is a toric hyperKähler manifold, consisting of a torus $\mathbf{T}^r$ fibered over a real $3r$ dimensional base. It is a simple matter to explicitly calculate the metric in each case, to discover that they coincide in the strong-coupling, infra-red limit $e^2 \to \infty$ [7, 10]. This is the statement of mirror symmetry in these theories.

It is instructive to examine the symmetries of each model. Theory A has a $\mathrm{U}(1)_F^{N-r} \times \mathrm{U}(1)_J^r$ global symmetry group. The F-currents are flavor symmetries acting on the chiral multiplets, while the J-currents act transitively on the dual photons σ, defined as $d\sigma = {}^\star F$,

$$\mathrm{U}(1)_J : \sigma \to \sigma + \alpha\,. \tag{2.5}$$

This is to be contrasted to the $\mathrm{U}(1)_F^r \times \mathrm{U}(1)_J^{N-r}$ global symmetry group of Theory B. The mapping is obvious: $F \leftrightarrow J$. In each case the $\mathrm{U}(1)^{N-r}$ factor is related to the mass and FI parameters of (2.4), while the $\mathrm{U}(1)^r$ factor acts on the $\mathbf{T}^r$ torus of the moduli space,

[2] Non-perturbative corrections, allowed by supersymmetry, are not present in abelian theories.

resulting in tri-holomorphic isometries of the metric. For certain choices of the charges, the flavor symmetry of either theory may be enhanced to a non-abelian group. In such circumstances, only the maximal torus is manifest in the mirror picture as a J-symmetry. Nevertheless, in the infra-red there is a quantum non-abelian symmetry enhancement and the symmetry groups of the two theories once again agree [7]. As well as its action on the global, tri-holomorphic symmetries, the mirror map also exchanges the $SU(2)_N \times SU(2)_R$ R-symmetry currents of the theories.

The above analysis leads us to view three dimensional mirror symmetry in the same light as Seiberg duality, valid only in the extreme infra-red. However, work by Kapustin and Strassler [42] suggests that in fact this need not be the case. They show that there exists a deformation of Theory B such that the metric on the Higgs branch coincides with the metric on the Coulomb branch of Theory A for all values of e^2:

$$\textbf{Theory B}' : \;\; U(1)^r \times \hat{U(1)}^N \text{ with } N \text{ hypermultiplets}$$

One may understand the deformation from Theory B to Theory B$'$ as a two-step process. One firstly gauges the $U(1)^r$ flavor symmetry of Theory B — it is the extra $\hat{U(1)}^r \subset \hat{U(1)}^N$ above. This is subsequently coupled via a Chern-Simons (CS) interaction to a further $U(1)^r$ symmetry group. These $U(1)^r$ fields have no further couplings to hypermultiplets, and their kinetic terms are normalized as $1/e^2$, the same as the coupling constants of Theory A. Examples of the resulting hyperKähler quotient construction were studied, for example, in [46]. The net effect is to squash, by an amount $1/e^2$, the asymptotic $\mathbf{T}^r$ fibers of the Higgs branch associated to the action of the flavor group. Since the action of the flavor group is tri-holomorphic, this squashing preserves the three complex structures on the Higgs branch. The resulting metric coincides with the Coulomb branch metric of Theory A at finite gauge coupling.

To fill in the details, the $\hat{U(1)}^N$ naturally splits into $r + (N-r)$ abelian gauge fields, under which the hypermultiplets have charge $(R_i^a, \hat{R}_i^p)$. The hypermultiplets are neutral under $U(1)^r$. These latter fields couple only to $\hat{U(1)}^r$ via a CS-coupling,

$$R_i^a R_i^b \; A^a \wedge \hat{F}^b \tag{2.6}$$

together with further terms required by supersymmetry (see below). The coupling constants e^2 of Theory A now play the role of coupling constants for the $U(1)^r$ of Theory B, while those of $\hat{U(1)}^N$ are sent to infinity (they may also be made finite if we further modify Theory A). Let us examine the vacuum moduli space of Theory B$'$,

$$\begin{aligned} V_{B'} = & \sum_{a=1}^{r} e_a^2 \left(R_i^a R_i^b \hat{\phi}^b\right)^2 + \sum_{a=1}^{r} \hat{e}_a^2 \left(R_i^a w_i^\dagger \vec{\tau} w_i + R_i^a R_i^b \phi^b\right)^2 + \\ & + \sum_{p=1}^{N-r} \hat{e}_p^2 \left(\hat{R}_i^p w_i^\dagger \vec{\tau} w_i + \hat{R}_i^p m_i\right)^2 + \sum_{i=1}^{N} \left(R_i^a \hat{\phi}^a + \hat{R}_i^p \hat{\phi}^p\right)^2 w_i^\dagger w_i \end{aligned} \tag{2.7}$$

where $\vec{\tau}$ are the Pauli matrices. The presence of ϕ and $\hat{\phi}$ in the D-terms is a consequence of the supersymmetric completion of the CS-coupling above. The Higgs branch of this theory is parameterized by w_i and ϕ_a, together with the r dual photons σ^a arising from $\mathrm{U}(1)^r$. The D-terms above provide $3N$ constraints which are moment maps for the $\hat{\mathrm{U}(1)}^N$ gauge orbits, the action of which includes a translation of the dual photons,

$$\begin{aligned} w_i &\to \exp\left(iR_i^b\hat{\alpha}_b + i\hat{R}_i^p\hat{\alpha}_p\right) w_i \\ \sigma^a &\to \sigma^a + R_i^a R_i^b \hat{\alpha}^b \end{aligned} \tag{2.8}$$

It is useful to examine how Theory B′ reduces to Theory B in the limit $e^2 \to \infty$. This allows the vector multiplet scalars ϕ to fluctuate unconstrained by a kinetic term. The ϕ's then appear only in the second term in (2.7) and their role is simply to remove this constraint. As for the corresponding $\hat{\mathrm{U}(1)}^r$ gauge action, this may be absorbed by its action on the dual photons (2.8), leaving the w_i to be constrained only by $(N-r)$ D-terms, and the corresponding gauge action $\hat{\mathrm{U}(1)}^{N-r}$. Note that, for certain non-minimal choices of charge R, there may remain a discrete remnant of the $\hat{\mathrm{U}(1)}^r$ gauge symmetry.

2.1 Deforming mirrors without cracking them

The agreement of the metrics — and hence the two-derivative terms in the low-energy expansion — at all values of the coupling constants suggests a view of mirror symmetry radically different from that first envisaged. Rather than being reminiscent of Seiberg duality, Theory A and Theory B′, may be thought of as two different descriptions of the same physics *at all energy scales*. This is the conjecture of Kapustin and Strassler [42]. Of course, this is a much stronger statement than mere agreement of the metrics on the vacuum moduli space, but nonetheless it has survived at least one non-trivial test [47]. For the purpose of this paper, we shall assume the validity of this conjecture, and will provide evidence that the conclusions we derive from this do indeed hold.

An important corollary of the Kapustin-Strassler conjecture is that it may be possible to deform the two theories, breaking supersymmetry but preserving their equivalence. This procedure was described for breaking to $\mathcal{N}=2$ Maxwell-Higgs theories in [43, 44] and to $\mathcal{N}=2$ Maxwell-Chern-Simons-Higgs theories in [45, 48]. One may attempt to be yet braver, and deform to theories with $\mathcal{N}=1$ supersymmetry. In this case we lose much control over our theories, since phase transitions abound. Nevertheless, we proceed blindly. In following sections, we shall re-derive some of these mirrors from string theory methods, giving further evidence of their validity.

Let us explain in more detail the techniques we use, starting with the theories without CS couplings. Suppose we gauge a flavor symmetry of Theory B′. The mirror deformation is to gauge the corresponding J-symmetry of Theory A. This is achieved via a CS coupling. This coupling requires the newly introduced dual photon to transform transitively under a gauge symmetry and, in the strong coupling limit, effectively removes this gauge symmetry. In this manner, gauging a flavor symmetry, say of Theory B′, is mirror to un-gauging a symmetry of Theory A. Moreover, since this procedure commutes with the strong coupling limit (at least in the classical lagrangian) we quote only the simpler mirror theories A and B: it is a trivial matter to re-derive the all-scale mirror "Theory B′ " following the prescription described above.

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If we perform this procedure in an $\mathcal{N}=2$ invariant manner — as first described in [43] — we arrive at the following mirror pairs,

$$\begin{aligned} &\textbf{Theory A}: \;\; \mathrm{U}(1)^r \text{ with } k \text{ neutral chirals and } N \text{ charged hypermultiplets} \\ &\textbf{Theory B}: \;\; \hat{\mathrm{U}(1)}^{N-r} \text{ with } N-k \text{ neutral chirals and } N \text{ charged hypermultiplets} \end{aligned}$$

where the gauge fields all lie within $\mathcal{N}=2$ vector multiplets. The charges of the hypermultiplets in the two theories, R_i^a and $\hat{R}_i^p$, are once again related through (2.2). The neutral chiral multiplets of Theory A, which we denote as Ψ_α, $\alpha=1,\ldots,k$, couple to the hypermultiplets, containing Q_i and $\tilde{Q}_i$, via the familiar gauge invariant cubic superpotential,

$$\mathcal{W} = S_i^\alpha \tilde{Q}_i \Psi_\alpha Q_i \tag{2.9}$$

with Yukawa couplings S_i^α. Theory B has analagous interactions with coupling constants $\hat{S}_i^\rho$, $\rho=1,\ldots,N-k$. These obey the mirror map,

$$\sum_{i=1}^{N} S_i^\alpha \hat{S}_i^\rho = 0 \qquad \forall\ \alpha,\rho\,. \tag{2.10}$$

All four matrices R, $\hat{R}$, S and $\hat{S}$ must be of maximal rank. While the above derivation of $\mathcal{N}=2$ mirrors relied on the mapping of the abelian symmetries, one may also arrive at them by use of the explicit operator mapping (2.3) as shown in [44].

We may now continue this procedure to $\mathcal{N}=1$ theories, with similar results. Before describing these results, let us recall some of the less familiar aspects of the $\mathcal{N}=1$ (two supercharge) theories. The $\mathcal{N}=2$ vector multiplet decomposes into an $\mathcal{N}=1$ vector multiplet and an $\mathcal{N}=1$ scalar multiplet. The former contains the gauge field and a single Majorana fermion. $\mathcal{N}=1$ gauge multiplets in three-dimensions contain no auxiliary fields, and all potential terms are therefore associated to scalar multiplets. These may be combined into a real scalar superfield which is a function of a single Majorana superspace coordinate (see [49, 19] for further details),

$$\Phi = \phi + \theta\psi - \theta^2 D \tag{2.11}$$

where ϕ is a real scalar, ψ is a Majorana spinor, and D is a real auxiliary field. Each of these fields has a natural complexification, resulting in a complex scalar superfield Q. This is nothing more than the familiar chiral superfield of $\mathcal{N}=2$ theories.

Since $\mathcal{N}=1$ supersymmetry in three dimensions provides no holomorphic luxuries, interactions between scalar superfields are written in terms of a *real* superpotential,

$$\int d^2\theta\ f(\Phi^a) = \frac{\partial f}{\partial\phi^a} D^a + \frac{\partial^2 f}{\partial\phi^a\partial\phi^b}\psi^a\psi^b\,. \tag{2.12}$$

With these conventions in mind, we can state our $\mathcal{N}=1$ mirror pairs. They are,

$$\begin{aligned} &\textbf{Theory A}: \;\; \mathrm{U}(1)^r \text{ with } k \text{ scalar and } N \text{ hypermultiplets} \\ &\textbf{Theory B}: \;\; \hat{\mathrm{U}(1)}^{N-r} \text{ with } (3N-k) \text{ scalar and } N \text{ hypermultiplets} \end{aligned}$$

where the gauge fields, this time, live in $\mathcal{N}=1$ vector multiplets and, by hypermultiplet, we mean the full $\mathcal{N}=4$ matter multiplet, each of which contains a doublet of complex scalar superfields. We write,

$$W = \begin{pmatrix} Q \\ \tilde{Q}^\dagger \end{pmatrix}. \tag{2.13}$$

As in previous cases, the coupling of these hypermultiplets to the gauge fields is through the charges R and $\hat{R}$ satisfying (2.2). The scalar multiplets couple only to the hypermultiplets through Yukawa couplings. For Theory A, the real superpotential is

$$f = \sum_{i,\alpha} W_i^\dagger \vec{\tau}\, W_i \cdot \vec{T}_i^\alpha\, \Phi_\alpha \tag{2.14}$$

$\alpha = 1, \ldots, k$. As before, $\vec{\tau}$ are Pauli matrices, and the couplings are determined by the triplet of $k \times N$ matrices, T. The Yukawa couplings for Theory B are of the same form, only now fixed by the triplet of $(3N-k)\times N$ matrices $\hat{T}$. They must satisfy,

$$\sum_{i=1}^{N} \vec{T}_i{}^\alpha \cdot \vec{\hat{T}}_i{}^\rho = 0 \qquad \forall\ \alpha, \rho \tag{2.15}$$

This is one of the main results of the paper and we shall employ it extensively in applications to D2-brane probes. As we have discussed above, the mirror symmetry of $\mathcal{N}=1$ three-dimensional gauge theories discussed here is conjectural, and difficult to prove. Nevertheless, we shall see later that this conjecture does yield the correct results in places where we can test it. For completeness, we should also point out that these are not the most general $\mathcal{N}=1$ mirrors, although they are all we need for the purposes of this paper. In the Appendix we describe further $\mathcal{N}=1$ mirrors with CS couplings.

Before proceeding, a comment on notation: as we have seen above, in this paper we shall consider $\mathcal{N}=1$ supersymmetric theories which include matter multiplets more usually found in theories with higher supersymmetry. We shall therefore refer to $\mathcal{N}=1,2$ and 4 matter multiplets as scalar, chiral and hyper-multiplets respectively. Throughout this paper, the following symbols will be used to denote various scalar fields:

- ϕ Real scalar field
- σ Real, periodic scalar, dual to the photon
- ψ Complex scalar field, neutral under the gauge group
- q Complex scalar field, charged under the gauge group
- w Doublet of complex scalar fields

The real scalar field ϕ may be found in either a scalar multiplet, or an $\mathcal{N}=2$ vector multiplet. Similarly, the complex scalar ψ lives in either a chiral multiplet, or an $\mathcal{N}=4$ vector multiplet. The complex scalar q arises in either a chiral or hypermultiplet, while the doublet w exclusively resides within a hypermultiplet. We will often write this explicitly as $w^\dagger = (q^\dagger, \tilde{q})$.

2.2 $\mathcal{N} = 1$ mirrors from IIB brane models

A large subclass of the mirror pairs decribed above have an alternative derivation in terms of Hanany-Witten type brane configurations in IIB string theory [9, 16]. We can construct many abelian $\mathcal{N} = 1$ field theories by considering D3-branes strung between various combinations of D5 and NS5-branes, lying in the directions,

$$\begin{array}{rl} D3 & 126 \\ NS5 & 12345 \\ NS5' & 12389 \\ D5 & 12348 \end{array}$$

An example of this type was discussed by Gremm and Katz in [16]. Note that, although

Configuration	Angles	Condition	SUSY	second 5-brane
1	θ_4	$\theta_4 = 0$	$\mathcal{N} = 4$	NS5 (12345)
2(i)	θ_2, θ_3	$\theta_2 = \theta_3$	$\mathcal{N} = 2$	NS5 $(123[48]_{\theta_2}[59]_{\theta_3})$
2(ii)	θ_3, θ_4	$\theta_3 = \theta_4$	$\mathcal{N} = 2$	$(p, q)5$ $(1234[59]_{\theta_3})$
3(i)	θ_1, θ_2, θ_3	$\theta_3 = \theta_1 + \theta_2$	$\mathcal{N} = 1$	NS5 $(12[37]_{\theta_1}[48]_{\theta_2}[59]_{\theta_3})$
3(ii)	θ_2, θ_3, θ_4	$\theta_3 = \theta_2 + \theta_4$	$\mathcal{N} = 1$	$(p, q)5$ $(123[48]_{\theta_2}[59]_{\theta_3})$
4(i)	θ_1, θ_2, θ_3, θ_4	$\theta_4 = \theta_1 + \theta_2 + \theta_3$	$\mathcal{N} = 1$	$(p, q)5$ $(12[37]_{\theta_1}[48]_{\theta_2}[59]_{\theta_3})$
4(ii)	θ_1, θ_2, θ_3, θ_4	$\theta_1 = -\theta_2$, $\theta_3 = \theta_4$	$\mathcal{N} = 2$	$(p, q)5$ $(12[37]_{\theta_1}[48]_{\theta_2}[59]_{\theta_3})$
4(iii)	θ_1, θ_2, θ_3, θ_4	$\theta_1 = \theta_2 = \theta_3 = \theta_4$	$\mathcal{N} = 3$	$(p, q)5$ $(12[37]_{\theta_1}[48]_{\theta_2}[59]_{\theta_3})$

Table 2: Supersymmetric five-brane configurations in IIB theory.

we deal exclusively with abelian mirrors, this brane set-up suggests many non-abelian mirror pairs. Here we extend this class of IIB brane models to more general configurations of five-branes. In fact, all models discussed by Gremm and Katz [16] have one massless scalar field, corresponding to D3-brane motion in the x^3 direction (common to all of the five-branes involved). More general five-brane configurations which lead to $\mathcal{N} = 1$ field theory on a D3-brane were classified in [50, 51] and nicely summarized in [52]. For completeness we (shamelessly) reproduce here a table from [51] wherein all possible five-brane configurations, together with the amount of supersymmetry on D3-branes stretched between them, are cataloged. Apart from its charge, each (p, q)-fivebrane is specified by its orientation in $x^3 - x^7$, $x^4 - x^8$, and $x^5 - x^9$ planes. We denote the corresponding angles by θ_1, θ_2, and θ_3, and refer to such a five-brane as:

$$(p, q)5 \qquad 12[37]_{\theta_1}[48]_{\theta_2}[59]_{\theta_3} \,. \tag{2.16}$$

As in [52], it will be convenient for us to label the charge of the 5-brane also by an angle:

$$\tan\theta_4 = \frac{p}{q} \,. \tag{2.17}$$

The models discussed by Gremm and Katz [16] correspond to case $3(ii)$. Here we consider examples of other models which will illustrate certain points.

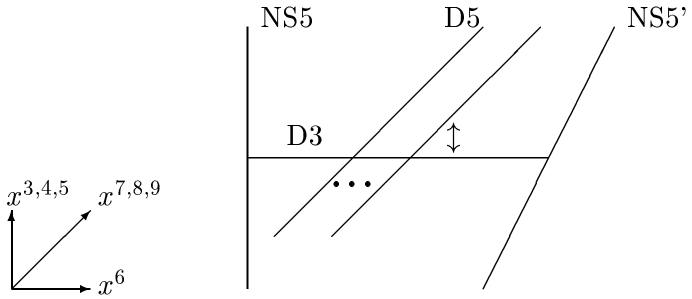

Figure 2: A D-brane model of $\mathcal{N} = 1$ theory in IIB string theory with NS5'-brane of type $3(i)$.

Example 1. We start with $3(i)$ model and a single U(1) gauge group. Apart from the first NS5-brane in directions $x^1, \ldots, x^5$, the configuration involves an NS5$'$-brane and N D5-branes as drawn in figure 2, and oriented as follows:

$$\begin{array}{rl} D3 & 126 \\ NS5 & 12345 \\ NS5' & 12[37]_{\theta_1}[48]_{\theta_2}[59]_{\theta_1+\theta_2} \\ D5 & 12789 \end{array}$$

Note, that if one of the angles θ_1, θ_2, or $\theta_3 = \theta_1 + \theta_2$ is equal to zero, then supersymmetry is enhanced to $\mathcal{N} = 2$. Below we assume a generic situation when this doesn't happen. The position of the D3-brane stretched between the NS5 and NS5$'$-branes is then fixed, so that the only massless mode is a U(1) gauge field. There are also N hypermultiplets coming from strings stretched between D3-brane and D5-branes.

As well as the massless modes, there are also some low-energy massive modes. The position of the D3-brane in x^3, x^4 and x^5 direction yields a triplet of scalars $\vec{\phi}$, whose mass $\vec{M}$ is determined by θ_1 and θ_2. For small angles, $M_1 \sim \tan\theta_1$, etc. Similarly, the position of the D5-branes in these same directions results in a triplet of mass parameters $\vec{m}_i$ which vanish when the D5 and D3-branes coincide. Summarizing, we have:

Theory A: U(1) with 3 massive scalar and N hypermultiplets

The scalar potential of this theory is given by,

$$V_A = \sum_{c=1}^{3} \left(\sum_{i=1}^{N} w_i^\dagger \tau_c w_i - M_c \phi_c \right)^2 + \sum_{i=1}^{N} |\vec{\phi} - \vec{m}_i|^2 w_i^\dagger w_i \,. \tag{2.18}$$

If hypermultiplets are massless, this theory has a classical Higgs branch of real dimension $(4N - 4)$. It corresponds to breaking the D3-brane into $N + 1$ segments, each stretched between adjacent D5-branes and free to move in directions x^7, x^8, and x^9 and with no constraint on the Wilson line A_6. At the origin of the Higgs branch there is a singularity at which the photon becomes massless. Emanating from this is the one-dimensional Coulomb branch, which is simply $\mathbf{S}^1$.

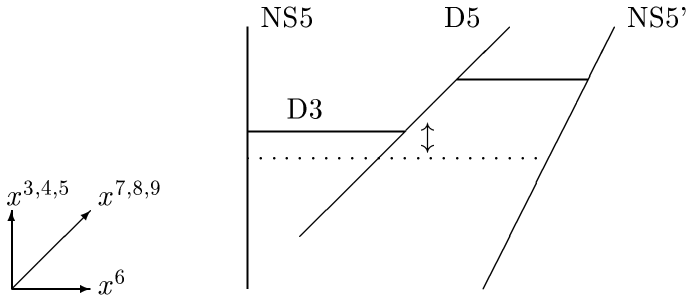

Figure 3: "Higgs branch" of a U(1) gauge theory with one massive hypermultiplet.

The Higgs branch remains if the hypermultiplets acquire an identical mass $\vec{m}_i = \vec{m}$. This can be seen from the string picture as the left-most segment moves in directions x^3, x^4, x^5 along the NS5-brane, while the rightmost segment has to move in directions x^7, x^8, x^9 along D5-brane, and in the same time also in directions x^3, x^4, x^5 along the NS5$'$-brane. This deformation is sketched in figure 3.

From the field theory perspective, we see that the mass parameter becomes promoted to a FI parameter upon setting $\vec{\phi} = \vec{m}$. Topologically, the Higgs branch is $T^\star\mathbb{CP}^{N-1}$, where the size of the zero section is determined by $\vec{M}$ and $\vec{m}$. The singularity at the origin of the Higgs branch is removed for $\vec{m} \neq 0$, but nevertheless the Coulomb branch remains: it is simply separated from the Higgs branch in field space. The fact that the vacuum moduli space is not connected implies that this theory contains domain walls interpolating between the Higgs and Coulomb phases.

The mirror theory can be obtained by S-duality in IIB string theory, which maps D5-branes into NS5-branes and vice versa. Specifically, we find:

Theory B: $\mathrm{U}(1)^{N-1}$ with $3(N-1)$ scalar and N hypermultiplets

This has the same matter content as the mirror theory derived by field theory means in the previous section. Moreover, the couplings may be easily read from the brane picture, and again agree with the field theory analysis. Each vector multiplet pairs up with 3 scalar multiplets to act essentially as a $\mathcal{N} = 4$ vector multiplet. These couple in an $\mathcal{N} = 4$ invariant fashion to all but the final hypermultiplet. The Coulomb and Higgs branches are $4(N-1)$ and 1 dimensional respectively.

Example 2. Let us now turn to an example that corresponds to a G_2 manifold that we discuss in section 4.4. It is an elliptic model (meaning the x^6 direction is compactified) of type 4i) from the table 2. The brane configuration has three five-branes of the same type, oriented to be mutually orthogonal. We start with,

$$\begin{aligned} D5_1 &: \ 12345 \\ D5_2 &: \ 12389 \\ D5_3 &: \ 12479 \\ D3 &: \ 126 \end{aligned} \tag{2.19}$$

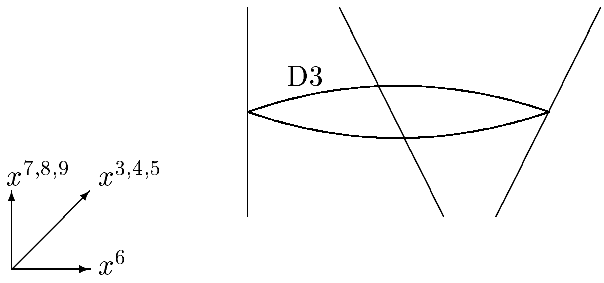

Figure 4: IIB Brane model for Example 2 with orthogonal 5-branes.

This is drawn in figure 4. The theory on the D3-brane has $\mathcal{N}=1$ supersymmetry and has the (interacting) massless field content,

Theory A: U(1) with 6 scalar and 3 hypermultiplets

where the real scalars ϕ_α, $\alpha = 1,\ldots,6$, denote the fields corresponding to D3-brane motion along $x^{3,4,5,7,8,9}$. These are coupled to the hypermultiplets through a superpotential of the form (2.14). To avoid writing out a triplet of 3×6 matrices, we may write the superpotential as,

$$f = \sum_{i=1}^{3} \vec{A}_i \cdot W_i^\dagger \vec{\tau} W_i\,, \tag{2.20}$$

where the triplets $\vec{A}_i$ are suitable combinations of the Φ_α,

$$\vec{A}_1 = (\Phi_7,\Phi_8,\Phi_9)\,, \qquad \vec{A}_2 = (\Phi_7,\Phi_4,\Phi_5)\,, \qquad \vec{A}_3 = (\Phi_3,\Phi_8,\Phi_5)\,. \tag{2.21}$$

This theory has a seven dimensional Coulomb branch, parameterized by the six scalars ϕ_α and the dual photon (hence the relevance to G_2 manifolds as we shall see). To read off the mirror theory, we perform a single S-duality to get,

$$\begin{aligned} NS5_1 &: \ 12345 \\ NS5_2 &: \ 12389 \\ NS5_3 &: \ 12479 \\ D3 &: \ 126 \end{aligned}$$

Since each pair of NS5-branes are mutually orthogonal, we may read off the theory on D3-brane world-volume by following the the standard rules for theories with $\mathcal{N}=2$ supersymmetry [5], treating each pair of NS5-branes in turn. It is simple to read off the massless fields from the brane set-up. We have,

Theory B: $\mathrm{U}(1)^2$ with 3 scalar and 3 hypermultiplets

where we have ignored the overall, free U(1) gauge symmetry. The hypermultiplets may be taken to have charges $(+1,-1,0)$ and $(0,-1,+1)$ under the gauge group. More subtle are the interactions. We expect two types of interactions; the usual Yukawa couplings of scalar multiplets to hypermultiplets, and the quartic superpotential couplings present in the $\mathcal{N}=2$ theories [53, 5]. Let us start with the Yukawa couplings. Recall that for a theory with $\mathcal{N}=4$ supersymmetry, the D- and F-terms are unified in a triplet of auxiliary fields which rotate into each other under the $\mathrm{SU}(2)_R$ R-symmetry. However, this symmetry is lost in the usual $\mathcal{N}=2$ superspace formulation of the theory. It may be made manifest if we work in $\mathcal{N}=1$ superspace, in which case the interaction terms are encoded in the real superpotential $\vec{\Phi}\cdot W^\dagger\vec{\tau}W$. One usually employs the notation that the D-term is proportional to τ^3, while the F-term is a complex combination of τ^1 and τ^2. In the present case, we have only "D-term" type interactions (only real scalar fields), but the rotation of the branes ensures that the scalar couples to a different combination for each hypermultiplet. Thus we find the Yukawa couplings,

$$\begin{aligned} f_{\mathrm{Yuk}} &= \Phi_1\left(W_1^\dagger\tau^3W_1 - W_2^\dagger\tau^3W_2\right) + \Phi_2\left(W_2^\dagger\tau^1W_2 - W_3^\dagger\tau^1W_3\right) + \\ &\quad +\Phi_3\left(W_3^\dagger\tau^2W_3 - W_1^\dagger\tau^2W_1\right). \end{aligned} \tag{2.22}$$

For the quartic superpotential, we assume that it arises from pairs on NS5-branes, in which case wemay follow the rules laid down in [5]. In the case where the D-term is proportional to τ^3 (for example, Φ_1 above), this quartic superpotential takes the form,

$$\tilde{Q}_1Q_1\tilde{Q}_2Q_2 + \mathrm{h.c.} = W_1^\dagger\tau^1W_1\,W_2^\dagger\tau^1W_2 - W_1^\dagger\tau^2W_1\,W_2^\dagger\tau^2W_2\,. \tag{2.23}$$

Hence, comparing with the indices in equation (2.22), we obtain,

$$\begin{aligned} f_4 &= W_1^\dagger\tau^1W_1\left(W_2^\dagger\tau^1W_2 - W_3^\dagger\tau^1W_3\right) + W_2^\dagger\tau^2W_2\left(W_3^\dagger\tau^2W_3 - W_1^\dagger\tau^2W_1\right) + \\ &\quad +W_3^\dagger\tau^3W_3\left(W_1^\dagger\tau^3W_1 - W_2^\dagger\tau^3W_2\right) \end{aligned} \tag{2.24}$$

As a simple check that this is the correct superpotential, note that it gives no further constraints on the w_i than (2.22) alone, ensuring that the Higgs branch of this model has the requisite dimension seven. The scalar potential arising from the superpotential $f = f_{\mathrm{Yuk}} + f_4$ yields the constraints,

$$\begin{aligned} |q_1|^2 - |\tilde{q}_1|^2 - |q_2|^2 + |\tilde{q}_2|^2 &= 0 \\ \mathrm{Re}(\tilde{q}_2q_2 - \tilde{q}_3q_3) &= 0 \\ \mathrm{Im}(\tilde{q}_3q_3 - \tilde{q}_1q_1) &= 0 \end{aligned}$$

These give 3 real constraints on the 12 real parameters w_i. After dividing by the $\mathrm{U}(1)^2$ gauge action, we arrive at the Higgs branch. The field content derived from the brane picture agrees with that derived in the previous section using field theory methods and, most importantly, the Higgs branches coincide. However, the brane theory also gave rise to the quartic superpotential (2.24). As discussed above, this higher dimension operator did not alter the vacuum moduli space, and the two techniques therefore agree. However, as we shall see in future examples, there are cases where the field content of the mirror theories derived through field theory techniques and brane constructions do not agree. In cases with a greater number of hypermultiplets, one often finds that the field theory mirror has a greater number of neutral scalars than the brane mirror. However, in all of these cases, the Higgs branch determined by the two theories is the same. Essentially, the extra constraints arising from the Yukawa couplings in the field theory context are exactly equal to the constraints arising from the quartic superpotential in the brane context. An example of this occurs in section 4.5. In fact, this same discrepancy between field theory methods and brane methods is also seen in mirror pairs with $\mathcal{N} = 2$ supersymmetry but, once again, the moduli spaces agree. Presumably in all these cases one may give masses to the extra scalar multiplets such that, after integrating them out, the quartic superpotentials are generated. Such points aside, we stress again that in this paper we are interested only in the vacuum moduli space and these agree in all cases.

3. SU(3) holonomy

In this section we discuss some of our techniques in the well-studied case of non-compact Calabi-Yau three-folds. Much of what we say here is not new, but we shall use the opportunity to stress the points which will play an important role in the forthcoming sections. We discuss separately the small resolution and deformation of the conifold and, in each case, attempt to work our way around figure 1. For the resolution these results are well-known and will be used extensively in later examples. However, for the deformation of the conifold to $T^\star \mathbf{S}^3$, the mirror probe theory is not well understood. We review the difficulties in this case.

The conifold under consideration is a three dimensional, non-compact, singular Calabi-Yau manifold X, defined by the complex equation,

$$xy - wz = 0\,. \tag{3.1}$$

As described in the introduction, we start with M-theory on X and and pick a $\mathbf{S}^1 \subset X$ on which to reduce, such that the resulting IIA spacetime is topologically $\mathbb{R}^6$. The sole remanant of the conifold is then two intersecting D6-branes with world-volumes [5, 6],

$$\begin{array}{c} D6\ 123456 \\ D6\ 123678 \end{array}$$

both of which lie at the same point in the x^9 direction. There exist two ways of smoothening out the singularity at the intersection point, $x = y = w = z = 0$. They are known as the small resolution, and the deformation. We deal with each in turn.

The small resolution. Changing the Kähler structure of the conifold results in the manifold $X \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$, which is topologically $\mathbb{R}^4 \times \mathbf{S}^2$. This deformation is non-normalizable: it appears as a parameter (as opposed to the VEV of a dynamical field) in the low-energy IIA action.

D-brane probes transverse to the resolved conifold have been well-studied in recent years — see for example, [53, 54, 55, 5, 6] — resulting in a full understanding of the circle of dualities discussed in the introduction. Let us review this chain of dualities, moving anti-clockwise around the circle of figure 1. We start by reducing to the L-picture. The equations (1.4) require the fixed point locus, L, to consist of two disconnected components, each of trivial topology. In other words, $L = \mathbb{C} \cup \mathbb{C}$ corresponding to two disconnected D6-branes which, by supersymmetry, are necessarily orthogonal. Together with the D2-brane probe, the world-volume directions span

$$\begin{array}{l} D2\ 12 \\ D6\ 123456 \\ D6\ 123678 \end{array}$$

While the world-volume directions of the D6-branes are identical to the singular case of the conifold, the singularity has been resolved by simply separating the D6-branes in the x^9 direction.

It is simple to write down the theory on the D2-brane. Since we shall be using this technique in later sections to derive more complicated mirror pairs, we spell out in detail how the spectrum arises. Firstly, the 2-2 strings result in the usual $\mathcal{N} = 8$ supersymmetry multiplet in three-dimensions, consisting of a U(1) gauge field and seven transverse scalars. The latter parameterise the motion of the D2-brane along x^m, $m = 3, \ldots, 9$. The U(1) gauge field may be dualised into a periodic scalar. It parameterizes the M-theory circle. The 2-6 strings supply a hypermultiplet for each D6-brane, whose couplings break supersymmetry to $\mathcal{N} = 2$. In the current situation, $(x^3 + ix^6)$ sits in a free chiral multiplet and shall be ignored in the following. The remaining four real scalars in the vector multiplet naturally form two neutral complex scalars,

$$\psi_1 = x^4 + ix^5\,, \qquad \psi_2 = x^7 + ix^8 \tag{3.2}$$

each of which is the lowest component of a chiral multiplet. Similarly, the U(1) gauge field combines with the real scalar $\phi = x^9$ to form a $\mathcal{N} = 2$ vector multiplet. The final theory therefore has matter content,

Theory A: U(1) with 2 neutral chirals and 2 charged hypermultiplets

The hypermultiplets come from 2-6 strings and have charge $(+1, -1)$ (note that the overall sign is a matter of convention). The scalar potential then is given by,

$$\begin{aligned} V_A = {} & e^2\left(|q_1|^2 - |q_2|^2 - |\tilde{q}_1|^2 + |\tilde{q}_2|^2\right)^2 + e^2|q_1\tilde{q}_1|^2 + e^2|q_2\tilde{q}_2|^2 + \\ & + \left(\phi^2 + |\psi_1|^2\right) w_1^\dagger w_1 + \left((\phi - m)^2 + |\psi_2|^2\right) w_2^\dagger w_2\,. \end{aligned} \tag{3.3}$$

In this picture, the Kähler class of the conifold is given by the real mass parameter m, corresponding to the separation of the D6-branes in the x^9-direction. This theory has a six dimensional Coulomb branch, parameterised by ϕ and ψ_i, together with the dual photon. By construction, this coincides with the resolved conifold. Note that, as stressed in the introduction, this is a statement only about the topology of the space; the metric on the Coulomb branch is not expected to coincide with the Ricci flat metric. When $m = 0$, there is a two dimensional Higgs branch, parameterised by the mesons $q_1\tilde{q}_2$, and $q_2\tilde{q}_1$. This corresponds to the D2-brane dissolved inside the intersection of the D6-branes.

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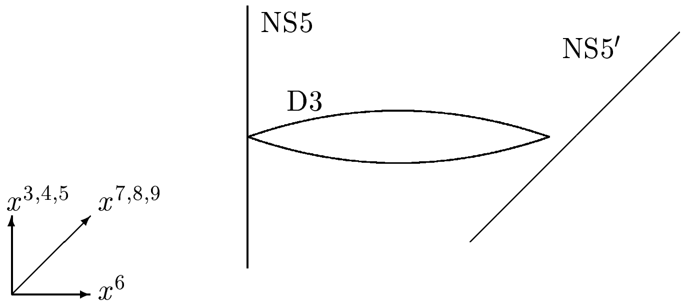

Figure 5: IIB Brane model for the deformed conifold.

Theory A has a $U(1)_J \times U(1)_1 \times U(1)_2 \times U(1)_F$ global symmetry group. The $U(1)_J$ factor acts only on the dual photon, while $U(1)_a$ rotates ψ_a, for $a = 1, 2$. In the strong coupling limit, we expect the $U(1)_J$ symmetry to be enhanced to SU(2), reflecting the full isometry group of the conifold. The hypermultiplet scalars are neutral under each of the first three U(1) factors. In contrast, the $U(1)_F$ symmetry does not act on the Coulomb branch, but rotates the hypermultiplet scalars: w_1 has charge $+1$, and w_2 charge -1. This global symmetry is the manifestation of the axial combination of gauge symmetries on the D6-brane which, in turn, arises from symmetries of the three-form C in M-theory.

To derive the mirror of Theory A, we continue to work our way around the circle in figure 1. Performing a T-duality in the x^6 direction, followed by an S-duality leads to the brane set-up shown in figure 5. The gauge theory on the D3-brane was first discussed in [12] (see also [5]) and reproduces the well-known linear sigma-model of the resolved conifold[3]

Theory B: U(1) + 2 charged hypermultiplets

The hypermultiplets both have charge +1 under the U(1) gauge field. There is a further free vector multiplet, parameterizing the motion of the D2-brane transverse to the conifold, and on the M-theory circle. This theory coincides with the mirror of Theory A derived in the previous section using field theory techniques [12, 44]. The scalar potential of this theory is given by,

$$V_B = e^2 \left(|q_1|^2 + |q_2|^2 - |\tilde{q}_1|^2 - |\tilde{q}_2|^2 - m\right)^2 + \phi^2 \left(w_1^\dagger w_1 + w_2^\dagger w_2\right). \tag{3.4}$$

The D-term, together with the action of the gauge group yields the resolved conifold of interest as the Higgs branch of the theory where the Kähler class is determined by the FI parameter. When $m = 0$, there is a two dimensional Coulomb branch, corresponding to D2-brane splitting into fractional branes. Quantum effects cause the Coulomb branch to split into two parts, parameterized asymptotically by $v_\pm = \exp(\pm\phi \pm i\sigma)$. Under mirror symmetry, $v_+ \to q_1\tilde{q}_2$, while $v_- \to q_2\tilde{q}_1$.

Finally, to close the circle of figure 1, we perform one further T-duality in the x^6 direction. The T-dual of intersecting NS5-brane configurations were discussed in [5, 6],

[3]The non-abelian world-volume theory on multiple D-brane probes is somewhat more involved, and includes a quartic superpotential [53]. This vanishes in the abelian case of interest.

and lead us back to IIA compactified on the resolved conifold. The theory on the D2-brane probe is once again Theory B, as may be derived through a process of partial resolutions of the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold [54]. This completes our journey around the resolved conifold.

The deformation. The singularity of the conifold may also be removed by deforming the defining equation to,

$$xy - wz = \rho \tag{3.5}$$

for some complex parameter ρ. This also changes the complex structure. The resulting Calabi-Yau manifold is $X \cong T^*\mathbf{S}^3$, and has topology $\mathbb{R}^3 \times \mathbf{S}^3$. As for the resolution, the deformation is non-normalizable in the IIA effective action. Let us try and repeat our discussion of the previous section, starting first with the L-picture. Using equations (1.4), we find that topologically $L \cong \mathbf{S}^1 \times \mathbb{R}$. Supersymmetry restrictions require L to be a complex curve (or, equivalently, a special lagrangian curve) in $\mathbb{C}^2$,

$$\psi_1\psi_2 = \rho\,. \tag{3.6}$$

From the U-dual perspective of NS5-branes, these deformations were dubbed brane diamonds in [55]. Our task is to reproduce this curve from the world-volume perspective of the D2-brane probe. Since asymptotically the locus L becomes the two orthogonal D6-branes described in (3.2) (now at the same x^9 position), we expect the field content to be the same as Theory A above; it remains to ask what further couplings the deformation parameter ρ induces in the theory.

Since this issue will play a role in later examples, let us spend some time examining the necessary quantum numbers of the deformation. We start with the M-theory picture. Recall that supersymmetry dictates that the moduli space of M-theory on the deformed conifold is parameterized by a hypermultiplet (in contrast to the resolved conifold which is parameterized by a five dimensional vector multiplet). The complex deformation ρ provides two of the four scalars in this hypermultiplet. A further scalar arises from the M-theory 3-form,

$$\theta = \int_{\mathbf{S}^3} C_3\,. \tag{3.7}$$

The global symmetries of M-theory on this background include not only the geometrical symmetries of X, but also a symmetry arising from the gauge symmetry of C, which act as translations on θ. For finite size $\mathbf{S}^3$, this symmetry is spontaneously broken. However, in the singular limit, the $\mathrm{U}(1)_C$ symmetry acting on θ_1 is restored. Indeed, we have already identified this symmetry in the probe theory of the previous section:

$$\mathrm{U}(1)_C \equiv \mathrm{U}(1)_F\,. \tag{3.8}$$

The deformation of the probe theory is therefore expected to break $\mathrm{U}(1)_F$.

Before determining the operator deformation of probe theory, let us examine how it arises from the D6-brane point of view. From this perspective, the deformation parameter ρ corresponds to a vacuum expectation value (VEV) for the hypermultiplet scalars arising from 6-6 strings localized at the intersection [55].[4] This VEV breaks the axial combination of the gauge symmetry on the D6-branes, in agreement with our comments about $\mathrm{U}(1)_C$ above. Moreover, this picture also suggests the correct D2-brane field theory deformation [55],

$$\mathcal{W} = M_1 \tilde{Q}_1 Q_2 + M_2 \tilde{Q}_2 Q_1 \tag{3.9}$$

which leads to the scalar potential,

$$\begin{aligned} V_A' &= e^2 \left(|q_1|^2 - |q_2|^2 - |\tilde{q}_1|^2 + |\tilde{q}_2|^2\right)^2 + e^2 |q_1 \tilde{q}_1|^2 + e^2 |q_2 \tilde{q}_2|^2 + \\ &\quad + \phi^2 w_1^\dagger w_1 + |\psi_1 \tilde{q}_1 + M_2 \tilde{q}_2|^2 + |\psi_1 q_1 + M_1 q_2|^2 + \\ &\quad + (\phi - |t|)^2 w_2^\dagger w_2 + |\psi_2 \tilde{q}_2 + M_1 \tilde{q}_1|^2 + |\psi_2 q_2 + M_2 q_1|^2 . \end{aligned}$$

This coupling leaves the $\mathrm{U}(1)_J \times \mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ symmetry of the probe theory unbroken, but is expected to alter the enhanced symmetry group at strong coupling. The $\mathrm{U}(1)_F$ symmetry is broken by this interaction, in agreement with expectations. Moreover, for $M_1 M_2 \neq 0$, it lifts the Higgs branch.

Finally, let us show that the interaction (3.9) does indeed correspond to the deformation (3.6). To see this, note that the hypermultiplets arising from 2-6 strings, which are expected to become massless on the curve (3.6). The two hypermultiplets of Theory A contain between them four Dirac fermions, whose mass matrix is,

$$\mathcal{M}_F = \begin{pmatrix} 0 & \psi_1^\dagger & 0 & M_2^\dagger \\ \psi_1 & 0 & M_1 & 0 \\ 0 & M_1^\dagger & 0 & \psi_2^\dagger \\ M_2 & 0 & \psi_2 & 0 \end{pmatrix} . \tag{3.10}$$

The curve L on which the D6-branes lies is determined by the zero locus of $\mathcal{M}_F$,

$$\det \mathcal{M}_F = |\psi_1 \psi_2 - M_1 M_2|^2 = 0 \tag{3.11}$$

which agrees with (3.6) for the choice of mass parameters $M_1 M_2 = \rho$.

This completes the D-brane probe in the L-picture. However, in attempting to move further around the circle of figure 1, we meet an obstacle. In particular, the theory on a D-brane probe transverse to X is, to our knowledge, undetermined. (For related work, see [55]). We may attempt to determine the mirror theory using purely field theoretical techniques [43]. The mirror of the operator (3.9) is

$$\Delta \mathcal{W} = M_1 V_+ + M_2 V_- \tag{3.12}$$

where $V_\pm$ are the chiral superfields parameterizing the quantum corrected Coulomb branch of Theory B. They are related to vortex creation operators [43] and are poorly understood. It would certainly be interesting check that the Higgs branch of Theory B with the above deformation does indeed reproduce the deformed conifold.

[4]The usual D-term constraints for hypermultiplets are suppressed by the ratio of the D6-brane world-volume to the volume of the intersection.

4. G_2 holonomy

In this section we consider M-theory on manifolds of G_2 holonomy. We start by recalling the condition on D6-branes for the preservation of $\mathcal{N} = 1$ supersymmetry in four dimensions [18]. Each of these IIA backgrounds lifts to M-theory compactified on a non-compact manifold X of G_2-holonomy. It is a simple matter to write down the theory on a D2-brane probe of these models. Employing our mirror pairs of three-dimensional gauge theories discussed in section 2, we find an algebraic quotient description of the manifold X as the Higgs branch. We stress that this procedure does not the provide G_2 holonomy metric on X. It does, however, give a simple construction of manifolds admitting a metric of G_2 holonomy, and may be employed as a linear sigma model for these purposes.

In the remainder of this section we give several examples of our construction, showing how it reproduces the cones over $\mathbb{CP}^3$ and $\mathrm{SU}(3)/\mathrm{U}(1)^2$ discussed in [3] as well as some of the results of [24]. We further use our technique to describe new G_2 manifolds arising from orthogonally intersecting D6-branes.

4.1 Intersecting special lagrangian planes

Consider M-theory compactified on a manifold X of G_2 holonomy. When the M-theory circle, $\mathbf{S}^1 \cong \mathrm{U}(1)$, is embedded in X, the resulting IIA string theory background consists of a six-manifold $X/\mathrm{U}(1)$, possibly with RR fluxes. Fixed points of the $\mathrm{U}(1)$ action lead to D6-branes. In the present case of a G_2 manifold, the condition on L to preserve four supercharges on the D6-brane is that L must describe a special lagrangian submanifold in $\mathbb{R}^6$.

In order to write this condition explicitly, let us define coordinates on the "interesting" part of the space, $\mathbb{R}^6$, parameterized by x^3, x^4, x^5, x^7, x^8, x^9. It is convenient to pair them into complex coordinates:

$$z_k = x^{2+k} + ix^{6+k}\,, \qquad k = 1,2,3 \tag{4.1}$$

and introduce a complex structure (a holomorphic SU(3) invariant 3-form):

$$\Omega = dz_1 \wedge dz_2 \wedge dz_3\,. \tag{4.2}$$

We also need a Kähler form on $\mathbb{R}^6 \cong \mathbb{C}^3$:

$$J = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3\,. \tag{4.3}$$

Now, the special lagrangian condition on 3-submanifold $L \subset \mathbb{R}^6$ implies two conditions: *a*) the restriction of J to L vanishes (then L is said to be lagrangian), and *b*) the restriction of $\mathrm{Im}(e^{i\gamma}\Omega)$ to L vanishes for some real phase $\gamma \in [0, 2\pi)$. Then,

$$\mathrm{Re}(e^{i\gamma}\Omega) \tag{4.4}$$

restricts on L as a volume form, and one says that L is *calibrated* with respect to $\mathrm{Re}(e^{i\gamma}\Omega)$ [56]. In particular, this condition defines an orientation of L:

$$\mathrm{Re}(e^{i\gamma}\Omega)|_L = \mathrm{vol}(L) > 0 \tag{4.5}$$

The simplest examples of a special lagrangian curve L is a collection of 3-planes linearly embedded into $\mathbb{R}^6 \cong X/\mathrm{U}(1)$. They correspond to *flat* D6-branes intersecting at a point, and lift to singular manifolds X of G_2 holonomy. Up to some obvious change of coordinates, such a special lagrangian 3-plane can be described by a choice of three angles θ_k, $k = 1, 2, 3$, corresponding to rotations of L in the $x^4 - x^7$, $x^5 - x^8$, $x^6 - x^9$ plane, respectively. Then, the special lagrangian conditions is simply [50]:

$$\theta_1 \pm \theta_2 \pm \theta_3 = 0\,, \qquad \mathrm{mod}\ 2\pi\,. \tag{4.6}$$

There are a few remarks in order here. First, note that none of the angles can be zero (mod 2π), for otherwise supersymmetry would be larger [50]. Secondly, a 'natural' choice of the orthogonal intersecting branes breaks all the supersymmetry; indeed it is T-dual to the well-known non-supersymmetric D0-D6 system. More generally, the above condition can never be satisfied if all $\theta_k = \pm\pi/2$.

Probe theory and a G_2 quotient construction. Consider a collection of N flat D6-branes, each of which has world-volume directions,

$$D6_i \qquad 123[47]_{\theta_1^i}[58]_{\theta_2^i}[69]_{\theta_3^i} \qquad i = 1, \ldots, N\,. \tag{4.7}$$

We probe this configuration with a D2-brane with spatial world-volume in the $x^1 - x^2$ plane. This breaks the supersymmetry of the D6-branes by a further half, resulting in $\mathcal{N} = 1$ supersymmetry in $d = (2 + 1)$ dimensions. For the singular case of intersecting, flat D6-branes, the theory on the D2-brane probe is simple to write down. The 2-2 strings give rise to the usual gauge field and seven scalars. Of these, there is one free $\mathcal{N} = 1$ scalar multiplet parameterizing motion in the x^3 direction common to all D6-branes. Further fields arise from the 2-6 strings. These give rise to N hypermultiplets which, as usual, we denote as $w_i^\dagger \equiv (q_i^\dagger, \tilde{q}_i)$. Thus we have the interacting $\mathcal{N} = 1$ supersymmetric theory on the probe as

Theory A: U(1) with 6 scalar multiplets and N hypermultiplets

where each hypermultiplet has charge $+1$ under the gauge field. The couplings of the hypermultiplets to the scalar multiplets are determined by the geometry of the D6-branes: each hypermultiplet couples minimally to the three scalar fields orthogonal to the corresponding D6-brane. We define the scalar fields $\phi_\alpha = x^{\alpha+3}$, $\alpha = 1, \ldots, 6$.

$$f = \sum_{i=1}^{N}\sum_{c=1}^{3}\sum_{\alpha=1}^{6} W_i^\dagger \tau^c W_i \cdot T_{c,i}^\alpha \phi_\alpha \tag{4.8}$$

where the couplings are determined by the triplet of matrices,

$$T_{c,i}^\alpha = -\sin\theta_c^i\,\phi_\alpha\,\delta_{c,\alpha} + \cos\theta_c^i\,\phi_\alpha\,\delta_{c,\alpha-3} \qquad c = 1, 2, 3\,. \tag{4.9}$$

The Coulomb branch of this theory, parameterized by the six real scalars ϕ_α, together with the dual photon σ, is a seven dimensional manifold X that admits a metric of G_2 holonomy. Since Theory A is of the class of theories discussed in section 2, we may simply write down the mirror theory whose Higgs branch is conjectured to give the G_2 manifold X,

Theory B: $\mathrm{U}(1)^{N-1}$ with $3(N-2)$ scalar and N hypermultiplets

The i^{th} gauge group acts on the i^{th} hypermultiplet with charge $+1$, and the $(i+1)^{\text{th}}$ hypermultiplet with charge -1. All other hypermultiplets are neutral. The Yukawa terms are of the same form as above,

$$f = \sum_{i=1}^{N} \sum_{c=1}^{3} \sum_{\rho=1}^{3N-6} W_i^\dagger \tau^c W_i \cdot \hat{T}^\rho_{c,i} \phi_\rho \,, \tag{4.10}$$

where, as in section 2, the triplet of coupling matrices are defined to satisfy,

$$\sum_{c,i} \hat{T}^\rho_{c,i} T^\alpha_{c,i} = 0 \qquad \forall\, \rho, \alpha \,. \tag{4.11}$$

The Higgs branch of this theory is parameterized by w_i, the $4N$ real scalars in the hypermultiplets, modulo the $(N-1)$ U(1) gauge orbits. To this we add the $3(N-2)$ real constraints coming from the Yukawa interactions,

$$\sum_{i,c} \hat{T}^\rho_{c,i} w_i^\dagger \tau^c w_i = 0 \qquad \rho = 1, \ldots, 3(N-2) \,. \tag{4.12}$$

This quotient construction yields a conical manifold which admits a metric of G_2 holonomy. In some cases the conical singularity may be (partially) resolved by adding constants to the right-hand side of (4.12). This blows up two-cycles and, in the IIA picture, corresponds to translating the D6-branes. Note that when the Yuakawa matrices $\hat{T}$ fall into suitable SU(2) triplets, the above method coincides with the toric hyperKähler quotient construction, supplemented by a further quotient by a tri-holomorphic isometry to yield a manifold of dimension seven. This is the construction discussed by Acharya and Witten [24]. However, in general, our charges differ. We now explain, in some detail, how this construction works in specific cases.

4.2 2 D6 Branes: the cone over $\mathbb{CP}^3$

We start with the simplest configuration, consisting of only two D6-branes. The natural 'symmetric' solution for the intersection of two D6-branes is when all the angles are equal:

$$\theta_1 = \theta_2 = \theta_3 = \frac{2\pi}{3} \,. \tag{4.13}$$

More precisely, by this we mean that D6-branes are represented by 3-planes inside $\mathbb{R}^6 \cong \mathbb{C}^3$, such that in each $\mathbb{C}$-plane they look like straight lines intersecting at $2\pi/3$ angle, see figure 6. Put differently, one can take this picture in one copy of $\mathbb{C}$, say in $x^3 - x^7$ plane, and tensor it three times.

As we shall see below, this D6-brane intersection lifts in M-theory to a cone on $\mathbb{CP}(3)$ with $G2$ holonomy metric [3]. A resolution of the conical singularity yields a smooth $G2$ manifold of the homotopy type [57]:

$$X \cong \mathbb{R}^3 \times \mathbf{S}^4 \,. \tag{4.14}$$

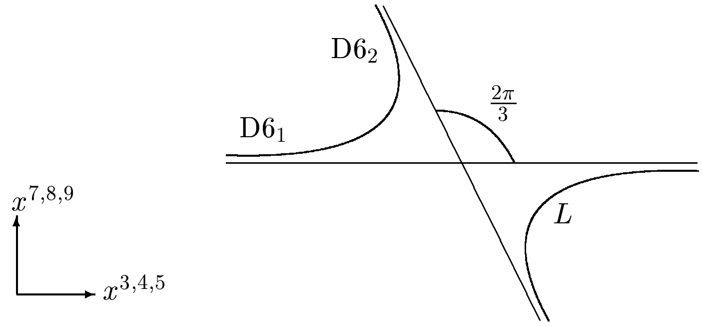

Figure 6: Intersection of special lagrangian D6-branes dual to M-theory on G_2 holonomy cone over $\mathbb{CP}(3)$. Resolution of the conical singularity corresponds to deforming intersecting D6-branes into a single smooth surface L.

As we described above, the proposed intersection of D6-branes has the right amount of supersymmetry, and as we explain below, it also has the right structure of global symmetries.

Before we go into the identification of symmetries, let us make a few general comments one should bear in mind. M-theory on $G2$-manifold X has certain global symmetries, some of which come from gauge symmetries of the C-field, while others are geometric symmetries of X itself. Let us denote the total group of these symmetries by K_X. If X develops a conical singularity the group of global symmetries is in general enhanced to a group of the corresponding symmetries of the six-dimensional base Y; we call it K_Y, *cf.* [3] For example, in the $Y = \mathbb{CP}^3$ model we are considering, the corresponding groups are:

$$K_X = \mathrm{Sp}(2) \times \mathbb{Z}_2 \tag{4.15}$$

and

$$K_Y = \mathrm{Sp}(2) \times \mathbb{Z}_2 \times \mathrm{U}(1)_C \tag{4.16}$$

where index C refers to the fact that $\mathrm{U}(1)_C$ comes from the gauge symmetries of the C-field. After we reduce this model to type IIA via compactification on a circle, the $\mathrm{U}(1)_C$ global symmetry can be understood as a gauge symmetry on D6-branes (remember, that the diagonal U(1) decouples) [3].

Turning to the geometric symmetries, after reduction from M-theory to type IIA, a geometric part of K_Y gets broken to a subgroup:

$$Sp(2) \to \mathrm{U}(1)_J \times \mathrm{SU}(2) \tag{4.17}$$

where $\mathrm{U}(1)_J$ is the isometry of the M-theory circle. It reappears in the probe picture as the dual photon. We will discuss this in more detail below. The SU(2) factor is to be identified with a geometric symmetry of the D6-brane configuration. Let us see how this appears. In the above notations for the spatial coordinates, one D6-brane can be described by *real* equations in $\mathbb{C}^3 \cong \mathbb{R}^6$:

$$D6_1 : \quad \mathrm{Im}(z_i) = 0\,. \tag{4.18}$$

One can easily check that the holomorphic 3-form defined above indeed restricts to a volume form on this special lagrangian plane, with a certain orientation.

The second D6-brane can be described by a similar set of linear equations:

$$D6_2 : \quad \mathrm{Im}(\omega z_i) = 0 \tag{4.19}$$

where $\omega = \exp(2\pi i/3)$. Again, the orientation is defined by (4.5). Written in terms of real coordinates x^m, these equations look like:

$$D6_1 : \quad x^7 = x^8 = x^9 = 0 \tag{4.20}$$

for the first D6-brane, and

$$D6_2 : \quad \frac{1}{2}x^7 + \frac{\sqrt{3}}{2}x^3 = \frac{1}{2}x^8 + \frac{\sqrt{3}}{2}x^4 = \frac{1}{2}x^9 + \frac{\sqrt{3}}{2}x^5 = 0 \tag{4.21}$$

for the second D6-brane.

These equations, defining the D6-brane locus L, are invariant under $\mathrm{SU}(2) \cong \mathrm{SO}(3)$ symmetry group, as expected for the $\mathbf{S}^4$ model at hand. To see this, let a be an element of SO(3), and introduce two 3-vectors:

$$\vec{\phi}_1 = \left(x^7, x^8, x^9\right)^T , \qquad \vec{\phi}_2 = \left(x^3, x^4, x^5\right)^T \tag{4.22}$$

Then, the SO(3) action is realized as follows:

$$\mathrm{SO}(3): \quad \vec{\phi}_1 \mapsto a \cdot \vec{\phi}_1 , \qquad \vec{\phi}_2 \mapsto a \cdot \vec{\phi}_2 . \tag{4.23}$$

Since SO(3) acts in the same way on both $\vec{\phi}_1$ and $\vec{\phi}_2$, the above equations manifestly remain invariant. Hence, the $\mathrm{SO}(3) \cong \mathrm{SU}(2)$ is a symmetry of our D6-brane configuration, in agreement with proposed relation to M-theory on $G2$ manifold X. Note that any other solution to the special lagrangian equations (4.6) would not have this property.

It remains to understand the supersymmetric deformation of this model away from the singular limit. The singularity of the manifold X may be resolved to have topology $\mathbb{R}^3 \times \mathbf{S}^4$. Comparing with equations (1.4), the D6-branes must deform to lie on a curve with topology,

$$L \cong \mathbb{R} \times \mathbf{S}^2 \tag{4.24}$$

asymptotic to a union of the special lagrangian 3-planes:

$$L_0 = \left\{ \vec{\phi}_1 = 0 \right\} \cup \left\{ \left(\vec{\phi}_1 + \sqrt{3}\vec{\phi}_2 \right) = 0 \right\} .$$

In fact, we may identify this curve L. The supersymmetry condition implies that L is special lagrangian submanifold in $\mathbb{C}^3 = \mathbb{R}^3 \times \mathbb{R}^3$, and $\mathrm{SU}(2) \cong \mathrm{SO}(3)$ symmetry implies that L is homogeneous in coordinates $\vec{\phi}_1$, $\vec{\phi}_2$:

$$\vec{\phi}_1 \cdot \vec{\phi}_2 = -|\vec{\phi}_1||\vec{\phi}_2| . \tag{4.25}$$

The special lagrangian condition gives one extra constraint [58]:

$$|\vec{\phi}_1| \left(3|\vec{\phi}_2|^2 - |\vec{\phi}_1|^2 \right) = \rho \tag{4.26}$$

that completely defines a 3-dimensional variety L, for every value of the real parameter ρ. Most importantly, for non-zero values of the deformation parameter ρ this equation defines a smooth surface with topology (4.24). In the singular limit $\rho \to 0$, we recover a configuration of two intersecting D6-branes described by L_0. As expected, the boundary of L (the same as boundary of L_0) is a union of two spheres:

$$F = \mathbf{S}^2 \cup \mathbf{S}^2 .$$

We therefore see that the moduli space of M-theory on X consists of a single parameter (suitably complexified). It was further shown in [3] that this deformation is $\mathbf{L}^2$-normalizable; the associated scalar has finite kinetic terms. This is in contrast to the conifold case discussed in the previous section. It would be interesting to understand this from the L-picture.

Probe theory. Having discussed the various properties of the L-picture, we now come to the theory on a D2-brane probe with spatial world-volume in the $x^1 - x^2$ plane. As discussed at the beginning of this section, the theory on the world-volume may be easily written down,

Theory A: U(1) with 6 scalar multiplets and 2 hypermultiplets

where the hypermultiplets arise from the 2-6 strings and are taken to both have charge +1 under the gauge field. The couplings between hypermultiplets and scalar multiplets are described in term of a real (non-holomorphic) superpotential,

$$f = \sum_{a,i=1}^{2} T_i^a \, W_i^\dagger \, \vec{\tau} \cdot \vec{\Phi}_a W_i \,, \tag{4.27}$$

where W_i for each $i = 1, 2$ is the hypermultiplet, expressed as a doublet of chiral multiplets; $\vec{\tau}$ are the Pauli matrices; and $\vec{\Phi}_a$, $a = 1, 2$ are each a triplet of scalar multiplets with lowest components $\vec{\phi}_a$. The Yukawa coupling matrix is given by,

$$T_i^a = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} , \tag{4.28}$$

where the hypermultiplet index $i = 1, 2$ labels the rows and the scalar multiplet index $a = 1, 2$ labels the columns. Note, in particular, the minus signs in the second row which distinguish a brane rotated by $2\pi/3$ from a brane rotated by $\pi/3$. The latter possibility is equivalent to replacing the brane with an anti-brane, and breaks supersymmetry. The bosonic sector of the probe theory does not notice the difference between these two possibilities. However, as we shall see momentarily, the minus signs are crucial for the fermionic sector. In terms of component fields, the superpotential leads to the scalar potential,

$$V = e^2 \left(w_1^\dagger \vec{\tau} w_1 + \frac{1}{2} w_2^\dagger \vec{\tau} w_2 \right)^2 + \frac{3}{4} e^2 \left(w_2^\dagger \vec{\tau} w_2 \right)^2 + w_1^\dagger w_1 |\vec{\phi}_1|^2 + \frac{1}{4} w_2^\dagger w_2 \left| \vec{\phi}_1 + \sqrt{3} \vec{\phi}_2 \right|^2 \tag{4.29}$$

One could further add FI terms to this theory. They lift the Coulomb branch of interest and correspond to background NS-NS B-fields, in the string theory picture. We set them to zero here. Any diagonal bare mass parameters can be absorbed by a shift of one of the scalars.

The symmetry group of the classical lagrangian is $U(1)_J \times SU(2) \times U(1)_F$. As usual the J-symmetry acts solely on the dual photon. The Coulomb branch scalars, $\vec{\phi}_1$ and $\vec{\phi}_2$ transform in the **3** of the SU(2), while the hypermultiplet scalars are singlets. Finally the $U(1)_F$ flavor symmetry acts on the hypermultiplets orthogonally to the gauge symmetry: we take w_1 to have charge $+1$ and w_2 charge -1.

Theory A has a moduli space of vacua in which the gauge group remains unbroken: it is the Coulomb branch. The D-terms of the potential (the first two terms in (4.29)) require us to set $w_1 = w_2 = 0$, ensuring that the scalar potential vanishes for any value of $\vec{\phi}_1$ and $\vec{\phi}_2$. The dual photon supplies the final moduli, bringing us to the requisite seven. As discussed in the introduction, the conservation of parity in this theory ensures that the Coulomb branch survives at the quantum level. Since the dual photon manifestly has a different origin to the other six scalars, the Coulomb branch has a natural decomposition into a $\mathbf{S}^1$ fiber over $\mathbb{R}^6$. The fiber degenerates at the positions of the D6-branes, as suggested by a one-loop computation in the gauge theory. At finite energies the Coulomb branch inherits the $U(1)_J \times SU(2)$ isometry from the field theory lagrangian. In the strong coupling limit, we expect this symmetry to be enhanced to SO(5).

The above probe theory describes the manifold X at the point where it develops a conical singularity. What deformation of Theory A smoothens out this singularity? As we shall see, the necessary change of the probe theory is more analogous to the deformation of the conifold, than to the small resolution of the conifold. Our clue is the $U(1)_F$ symmetry of the probe theory. This does not act as an isometry on the Coulomb branch, arises from a $U(1)_C$ C-field symmetry of M-theory on X. From the discussion of [3] we know that the $U(1)_C$ symmetry is generically broken. It is restored only when X develops a conical singularity. The situation is therefore very similar to the deformed conifold picture. To describe the probe theory on the resolved locus (4.26), we are looking for a deformation of Theory A which breaks the $U(1)_F$ flavor symmetry while simultaneously preserving the existence, not only of the Coulomb branch, but also of the SU(2) symmetry. We claim that the correct deformation is once again of the form (3.9) but, without the luxury of $\mathcal{N} = 2$ supersymmetry, we need not restrict ourselves to holomorphic superpotentials. However, the SU(2) symmetry does place severe constraints on the possible couplings. To see this, note that while the hypermultiplet scalars do not transform under this global symmetry, the same is not true of the hypermultiplet fermions. This is apparent if we examine the Yukawa couplings in Theory A. With supersymmetry broken to $\mathcal{N} = 1$, it is most natural to work with real Majorana rather than complex Dirac fermions. In $d = (2+1)$, each hypermultiplet contains four Majorana fermions, λ^p, $p = 1, 2, 3, 4$. The couplings to the vector multiplet scalars are,

$$\left(\lambda_1^p \vec{\phi}_1 \lambda_1^q - \lambda_2^p \left(\frac{1}{2}\vec{\phi}_1 + \frac{\sqrt{3}}{2}\vec{\phi}_2\right) \lambda_2^q\right) \cdot \left(\vec{V}_{pq} + i\vec{\eta}_{pq}\right), \tag{4.30}$$

where the $\vec{\eta}$ are the self-dual 4×4 anti-symmetric 't Hooft matrices, and the $\vec{V}$ are 4×4 symmetric matrices given by,

$$\begin{array}{lll} \eta_1 \;=\; i\tau^2 \otimes \tau^1 \,, & \eta^2 \;=\; -i\tau^2 \otimes \tau^3 \,, & \eta^3 \;=\; 1 \otimes i\tau^2 \\ V^1 \;=\; -\tau^1 \otimes \tau^3 \,, & V^2 \;=\; -\tau^1 \otimes \tau^1 \,, & V^3 \;=\; \tau^3 \otimes 1 \end{array}$$

which satisfy the relations

$$[\eta^i, \eta^j] = -2\epsilon^{ijk}\eta^k \,; \qquad [\eta^i, V^j] = -2\epsilon^{ijk}V^k \,; \qquad [V^i, V^j] = 2\epsilon^{ijk}\eta^k \,. \tag{4.31}$$

Note that the anti-symmetric $\vec{\eta}$ terms in the Yukawa coupling vanish in the abelian theory of interest. The relative minus sign between the two couplings follows from the superpotential couplings (4.28). Under the SU(2) symmetry, $\vec{\phi}_a$ transform in a triplet which implies the transformations of the fermions,

$$\mathrm{SU}(2) \;:\; \vec{\delta}\lambda_1^p = \vec{\eta}^p{}_q\lambda_1^q \,, \qquad \vec{\delta}\lambda_2^p = \vec{\eta}^p{}_q\lambda_2^q \,. \tag{4.32}$$

In these conventions the gauge and flavor transformations are implemented by use of the anti-symmetric, anti-self dual 't Hooft matrix, $\bar{\eta}_3 = -i\tau^3 \otimes \tau^2$, satisfying $[\bar{\eta}^3, \vec{\eta}] = [\bar{\eta}^3, \vec{V}] = 0$,

$$\begin{array}{ll} \mathrm{U}(1)_G \,:\, \delta_G\lambda_1^p \,=\, (\bar{\eta}^3)^p{}_q\lambda_1^q \,, & \delta_G\lambda_2^p \,=\, (\bar{\eta}^3)^p{}_q\lambda_2^q \\ \mathrm{U}(1)_F \,:\, \delta_F\lambda_1^p \,=\, (\bar{\eta}^3)^p{}_q\lambda_1^q \,, & \delta_F\lambda_2^p \,=\, -(\bar{\eta}^3)^p{}_q\lambda_2^q \end{array}$$

From these transformation laws, we may deduce the correct generalization of the coupling (3.9): it is a superpotential that results in the real fermion mass terms,

$$\lambda_1^p X_{pq} \lambda_2^q \,, \tag{4.33}$$

where X must satisfy $[X, \vec{\eta}] = [X, \bar{\eta}^3] = 0$ for the coupling to have the correct quantum numbers. The only solution to this equation is

$$X = M_1 \cdot \mathbf{1} + M_2 \cdot \bar{\eta}^3 \,, \tag{4.34}$$

where M_1 and M_2 are real parameters, and $\mathbf{1}$ is the unit 4×4 matrix. The hypermultiplet fermion mass matrix arising from the Yukawa coupling (4.30), together with this deformation is an 8×8 real, symmetric matrix $\mathcal{M}_F$ with determinant,

$$\det \mathcal{M}_F = \left(M^4 + M^2 \left(\vec{\phi}_1 + \sqrt{3}\vec{\phi}_2 \right) + \frac{1}{4}|\vec{\phi}_1|^2 \left| \vec{\phi}_1 + \sqrt{3}\vec{\phi}_2 \right|^2 \right)^2 \tag{4.35}$$

with $M^2 = M_1^2 + M_2^2$. The zero locus of the determinant in the full quantum theory is expected to reproduce the special lagrangian locus L given by equations (4.25) and (4.26). Our (rather conservative) hope is that the semi-classical analysis gives at least the right topology of L. At first sight it seems rather difficult to reproduce the two equations defining L from the root of an order eight polynomial. However, for the fermion mass matrix above, the equation $\det \mathcal{M}_F = 0$ only has solutions when ϕ_1 and ϕ_2 are anti-parallel,

$$\vec{\phi}_1 \cdot \vec{\phi}_2 = -|\vec{\phi}_1||\vec{\phi}_2| \tag{4.36}$$

and the roots then satisfy the further constraint

$$M^2 = |\vec{\phi}_1| \left(|\sqrt{3}|\vec{\phi}_2| - |\vec{\phi}_1| \right) \geq 0\,. \tag{4.37}$$

Comparing this locus with the special lagrangian curve L given in equations (4.25) and (4.26), we see that the curves do not precisely agree. As mentioned in the introduction, we have no reason to expect exact agreement. Indeed, the zero locus of the fermion mass matrix yields the one-loop correction to the degeneration of the dual photon fiber, and we have no reason to expect higher loop corrections to be negligible near the intersection of the D6-branes. Nevertheless, the topology of the locus derived from field theory does coincide with the special lagrangian manifold L. This is heartening.

Finally, we note that if the minus sign in (4.30) is replaced with a plus — corresponding to branes rotated by $\pi/3$, breaking supersymmetry — the fermion masses vanish on a locus with $\vec{\phi}_1$ and $\vec{\phi}_2$ parallel. This is not a special lagrangian deformation, reflecting the breaking of supersymmetry in that case.

IIB brane construction and the mirror theory. Let us now continue our way around figure 1, and derive the mirror theory whose Higgs branch yields the G_2 manifold X. We have presented two methods to derive the mirror theory: using field theory techniques and D-brane models. Here we use both. Firstly, from the discussion of section 2.1, it is simple to write down the field theory mirror,

Theory B: U(1) with 2 hypermultiplets.

where the hypermultiplets have charge $+1$ and -1 respectively. This theory has a seven real-dimensional Higgs branch $X = \mathbb{C}^4 / \mathrm{U}(1)$. Since $\mathbb{C}^4$ may be thought of as the cone over $\mathbf{S}^7$, and $\mathbf{S}^7/\,\mathrm{U}(1) \cong \mathbb{CP}^3$, the resulting Higgs branch is topologically the cone over Y, with

$$Y = \mathbb{CP}^3$$

in agreement with the claims made in this section. Note that this theory yields the singular, conical limit of X. The above procedure can also be performed for the smooth locus L given in (4.26), resulting in brane diamond configurations analogous to those of [55], but with lower supersymmetry. As with the deformed conifold case, we do not well understand the mirror theory whose Higgs branch yields the blown up G_2 manifold. Presumably, as in the conifold case, the mirror of the supersymmetric operator (4.33) is related to the vortex creation operator.

We can also derive this theory using brane configurations by T-dualizing to IIB along the x^6 direction, common to both D6-branes, and transverse to the D2-brane. If T-duality is performed in the singular limit of flat intersecting D6-branes, the resulting brane configuration consists of D5-branes, with a single D3-brane, each with world-volume directions,

$$\begin{array}{ll} D3 & 126 \\ D5_1 & 12789 \\ D5_2 & 12[37]_{\theta_1}[48]_{\theta_2}[59]_{\theta_3} \end{array}$$

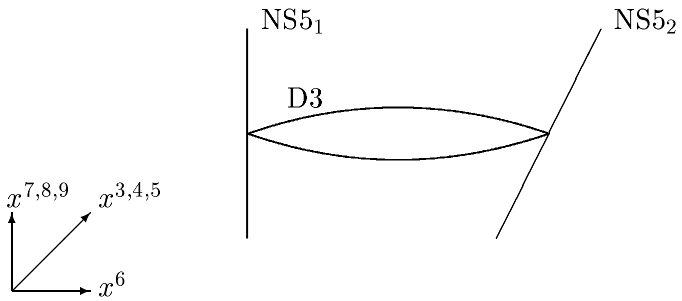

Figure 7: IIB Brane model for Theory B: U(1) with 2 hypermultiplets.

where

$$\theta_1 = \theta_2 = \theta_3 = \frac{2\pi}{3}\,. \tag{4.38}$$

A Wilson line for the flavor symmetry ensures the D5-branes are separated along the x^6 circle. In the low-energy limit, the theory on the D3-brane probe of this periodic ("elliptic") model coincides with Theory A described above. More interesting is the D3-brane world-volume theory upon performing an S-duality. Following Hanany and Witten [9], this should result in the mirror theory. The IIB brane configuration is depicted in figure 7. The world-volumes are,

$$\begin{aligned} D3 &\quad 126 \\ NS5_1 &\quad 12789 \\ NS5_2 &\quad 12[37]_{\theta_1}[48]_{\theta_2}[59]_{\theta_3} \end{aligned}$$

The theory on the D3-brane probe may be read from the brane picture. Each segment yields a U(1) vector field, as well as three scalar multiplets. Each of these scalar multiplets acquires a mass due to the relative orientation of the NS5-branes. Further, there exist hypermultiplets arising from strings stretched across each NS5-brane. These have charges $(+1,-1)$ and $(-1,+1)$, respectively, under the two gauge factors, ensuring that the diagonal U(1) decouples. The massless fields are simply those of Theory B above. Finally, note that the brane picture also has the possibility of a superpotential, arising from integrating out massive neutral scalars,

$$f = \sum_{a,i=1}^{2} \tilde{S}_i^a\, W_i^\dagger\, \vec{\tau} \cdot \vec{\Phi}_a W_i + \sum_{a=1}^{2} m_a \vec{\Phi}_a \cdot \vec{\Phi}_a\,, \tag{4.39}$$

where the scalars couple as

$$\tilde{S}_i^a = \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}. \tag{4.40}$$

By symmetry the scalar masses are $m_1 = -m_2$, ensuring that upon integrating out the scalar multiplets there is no remaining superpotential. At low-energies the resulting dynamics is therefore that of Theory B.

Generalizations: the cone over $\mathbb{WCP}^3$. There is an obvious generalization to this model in which we add further D6-branes, lying parallel to the ones already introduced. Consider the IIA background consisting of $p+q+2$ D6-branes with orientation,

$$\begin{array}{rl} D2 & 12 \\ (p+1)\times D6_1 & 126789 \\ (q+1)\times D6_2 & 126[37]_{\theta_1}[48]_{\theta_2}[59]_{\theta_3} \end{array}$$

where, as before

$$\theta_1 = \theta_2 = \theta_3 = \frac{2\pi}{3}\,. \tag{4.41}$$

We claim that this IIA brane configuration lifts to M-theory on the singular G_2 manifold X which is the cone over the weighted projective space $\mathbb{WCP}^3_{p,p,q,q}$. These manifolds were discussed in [3, 24] in the quest for G_2 compactifications yielding four dimensional chiral fermions. The manifold X has two co-dimension 4 singularities of ALE type: an A_p and an A_q. If M-theory is compactified on X, these singularities support a seven dimensional $\mathrm{SU}(p)$ and $\mathrm{SU}(q)$ gauge connection respectively. The intersection of this singularity supports a chiral fermion in the $(\mathbf{p},\mathbf{q})$ representation of $\mathrm{SU}(p)\times \mathrm{SU}(q)$.

The theory on a D2-brane probe of this D6-brane configuration is,

Theory A: $\mathrm{U}(1)$ with 6 scalars and $N+2$ hypermultiplets

where $N=p+q$. All hypermultiplets have charge $+1$ under the the gauge fields. The 6 scalar multiplets pair up into two triplets as described in the previous section (4.27). The first $p+1$ hypermultiplets have Yukawa couplings with the first triplet, while the remaining $q+1$ chirals couple to both triplets with charges (4.28). The singularity where the D6-branes meet may be partially resolved by separating the D6-branes. From the probe perspective, this corresponds to adding mass terms for the hypermultiplets, and results in pq singularities of the type we met in the previous section. These may further be resolved by introducing operators in the probe theory of the form (4.33).

The simplest way to see that this background indeed lifts to the cone over $\mathbb{WCP}^3_{p,p,q,q}$ is to examine the mirror three-dimensional gauge theory. Again, we perform T- and S-dualities to type IIB, to find a configuration of N NS5-branes, and a single D3-brane. In this case we find a further subtlety: the theory on the D3-brane depends on the relative ordering of the NS5-branes around the circle. Here we concentrate on the simplest case where all branes with the same orientation are adjacent. (It would be interesting to examine whether exchanging brane positions results in Seiberg-like duality for these three-dimensional gauge theories [59]). The interacting $\mathcal{N}=1$ mirror theory may then simply be read from the brane picture:

Theory B: $\mathrm{U}(1)^{N+1}$ with $3N$ scalars and $N+2$ hypermultiplets

Notice that the number of gauge fields is one more than the number of scalar triplets. In fact, we may combine the scalar multiplets with N of the gauge fields into $\mathrm{U}(1)^N = \mathrm{U}(1)^p\times\mathrm{U}(1)^q$ $\mathcal{N}=4$ vector multiplets. These couple to the hypermultiplets in an $\mathcal{N}=4$

invariant fashion as follows: the first $p+1$ hypermultiplets are charged only under $\mathrm{U}(1)^p$, while the remaining $q+1$ multiplets are charged under $\mathrm{U}(1)^q$. The charges of each subset are determined by the quiver diagrams of $\mathrm{SU}(p+1)$ and $\mathrm{SU}(q+1)$ respectively (where in each case the overall, free, gauge field is ignored). Before taking into account the remaining $\mathcal{N}=1$ U(1) vector multiplet, the resulting Higgs branch is therefore the hyperKähler 8-manifold given by the direct product $\mathbb{C}^2/\mathbb{Z}_p \times \mathbb{C}^2/\mathbb{Z}_q$.

The extra U(1) action in Theory B reduces the Higgs branch to a seven dimensional manifold. It acts as a U(1) quotient of $\mathbb{C}^2/\mathbb{Z}_p \times \mathbb{C}^2/\mathbb{Z}_q$, which acts with charge $+1$ on the $(p+1)^{\mathrm{th}}$ hypermultiplet, and charge -1 on the $(p+2)^{\mathrm{th}}$, with all others neutral. This is precisely the construction [24] where manifolds admitting G_2 holonomy metrics were constructed by quotienting hyperKähler manifolds by a tri-holomorphic isometry. This particular example was considered in [24] where it was shown that the Higgs branch is indeed a cone over $\mathbb{WCP}^3_{p,p,q,q}$.

4.3 3 D6 branes: the cone over $\mathrm{SU}(3)/\mathrm{U}(1)^2$

We turn now to the second example of Atiyah and Witten [3], the cone over the flag manifold $Y = \mathrm{SU}(3)/\mathrm{U}(1)^2$. A resolution of this singular space yields a smooth G_2 manifold of homotopy type [57]

$$X \cong \mathbb{R}^3 \times \mathbb{CP}^2\,. \tag{4.42}$$

Atiyah and Witten [3] argue that the moduli space of M-theory compactified on this space contains three components, intersecting at a singular point and rotated by a triality symmetry. In the following we shall confirm this scenario using the D-brane probe theory.

There is a natural guess for the D6-brane intersection of the singular conical manifold X. This is based on observation that adding an extra D6-brane oriented at angle $4\pi/3$ to the previous example does not break supersymmetry further. Note, due to the special lagrangian condition (4.5) a similar configuration of D6-branes rotated by angle $2\pi/6$ would be non-supersymmetric; the D6-branes would have opposite orientation, i.e. correspond to anti-branes. Summarizing, we obtain a configuration of three intersecting D6-branes, which look like 3-planes intersecting at angles $2\pi/3$ in directions x^3-x^7, x^4-x^8, x^5-x^9, see figure 8.

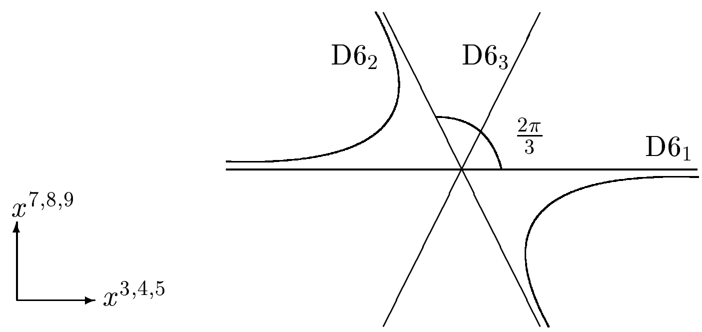

Figure 8: Intersection of special lagrangian D6-branes dual to M-theory on G_2 holonomy cone over $\mathrm{SU}(3)/\mathrm{U}(1)^2$. Resolution of the conical singularity corresponds to deforming intersecting D6-branes into a smooth surface L with two connected components.

There is a simple way to obtain this configuration, which makes it clear that the amount of unbroken supersymmetry is as in the previous example. Suppose, we start with a single D6-brane, say $D6_1$ in figure 8, in the flat ten-dimensional space-time. This configuration is 1/2 BPS. Now, let us orbifold this space by a $\mathbb{Z}_3$ group acting as follows:

$$\mathbb{Z}_3: \quad z_k \mapsto \omega z_k \,, \qquad k = 1,2,3 \tag{4.43}$$

where, as before, ω is denotes a cube root of unity, $\omega = \exp(2\pi i/3)$. Since this $\mathbb{Z}_3$ is a subgroup of SU(3), this orbifold preserves 1/4 of the original supersymmetry, i.e. the resulting configuration is 1/8-BPS (equivalent to $\mathcal{N} = 1$ supersymmetry in four dimensions). We can think of the resulting configuration as an intersection of three D6-branes on the covering space (which is again just $\mathbb{C}^3$), corresponding to the $\mathbb{Z}_3$ mirror images of the original $D6_1$ brane. Thus, we find the proposed configuration of D6-branes, shown on figure 8.

Explicitly, we can describe the i-th D6-brane by the set of linear equations:

$$\mathrm{Im}(\omega^{i-1} z_k) = 0, \quad \forall k = 1,2,3 \tag{4.44}$$

We can write these equations in terms of real coordinates x^m. The equations for the first two D6-branes are exactly the same as in the previous example:

$$D6_1: \quad x^7 = x^8 = x^9 = 0 \tag{4.45}$$

$$D6_2: \quad \frac{1}{2}x^7 + \frac{\sqrt{3}}{2}x^3 = \frac{1}{2}x^8 + \frac{\sqrt{3}}{2}x^4 = \frac{1}{2}x^9 + \frac{\sqrt{3}}{2}x^5 = 0 \tag{4.46}$$

and for the new $D6_3$ brane we have a similar set of three linear equations:

$$D6_3: \quad \frac{1}{2}x^7 - \frac{\sqrt{3}}{2}x^3 = \frac{1}{2}x^8 - \frac{\sqrt{3}}{2}x^4 = \frac{1}{2}x^9 - \frac{\sqrt{3}}{2}x^5 = 0\,. \tag{4.47}$$

Let us compare the symmetries of this D6-brane configuration with those expected from the quotient of the cone over the flag manifold $Y = \mathrm{SU}(3)/\,\mathrm{U}(1)^2$. The symmetry group of M-theory in the singular conical limit is,

$$K_Y = \mathrm{SU}(3) \times \mathrm{U}(1)^2_C\,. \tag{4.48}$$

The $\mathrm{U}(1)^2_C$ symmetry coincides with the (axial) gauge symmetries of the D6-branes, while the SU(3) geometrical symmetry is reduced upon squashing to

$$\mathrm{SU}(3) \to \mathrm{U}(1)_J \times \mathrm{SU}(2)\,. \tag{4.49}$$

As usual, the $\mathrm{U}(1)_J$ corresponds to the M-theory circle, while the SU(2) must survive as a geometrical symmetry of the IIA configuration. Indeed, our three intersecting D6-branes have precisely such an invariance:

$$\mathrm{SO}(3): \quad \vec{\phi}_1 \mapsto a \cdot \vec{\phi}_1\,, \qquad \vec{\phi}_2 \mapsto a \cdot \vec{\phi}_2\,. \tag{4.50}$$

We would now like to discuss the resolution of the singularity in this model. From equations (1.4), we expect the resolved locus resolved locus L to have $h_0(L) = h_2(\mathbb{C}\mathbf{P}^2) + 1 = 2$ and $H_2(L,\mathbb{Z}) \cong H_4(\mathbb{C}\mathbf{P}^2) \cong \mathbb{Z}$, so that topologically

$$L \cong \mathbb{R} \times S^2 \cup \mathbb{R}^3 \tag{4.51}$$

In the previous section we noticed that the intersecting $\mathrm{D6}_1$-brane and $\mathrm{D6}_2$-brane can be continuously deformed into a single smooth D6-brane described by the equations:

$$\vec{\phi}_1 \cdot \vec{\phi}_2 = -|\vec{\phi}_1||\vec{\phi}_2|, \qquad |\vec{\phi}_1| \cdot (3|\vec{\phi}_2|^2 - |\vec{\phi}_1|^2) = \rho \tag{4.52}$$

In fact, it is easy to see that the same deformation also resolves singularity in this model. Namely, it deforms three intersecting special Lagrangian planes into two components of the smooth locus L:

$$L = \Big\{ \vec{\phi}_1 \cdot \vec{\phi}_2 = -|\vec{\phi}_1||\vec{\phi}_2| \ ; \quad |\vec{\phi}_1|(3|\vec{\phi}_2|^2 - |\vec{\phi}_1|^2) = \rho \Big\} \cup \Big\{ |\vec{\phi}_1 - \sqrt{3}\vec{\phi}_2| = 0 \Big\} \tag{4.53}$$

The first connected component of L is what one finds from the deformation of $\mathrm{D6}_1$ and $\mathrm{D6}_2$, as in the previous section, whereas the second component is just the original plane $\mathrm{D6}_3$-brane. Note that equations, which parameterize the first and the second component of L have no common solutions. Deforming the intersection of two of the D6-branes creates a hole through which the third passes. Therefore, the above deformation completely removes the singularity, and we obtain a completely smooth D6-brane locus, diffeomorphic to the locus (4.51) as expected.

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Of course, instead of deforming the $\mathrm{D6}_1$ and $\mathrm{D6}_2$ branes, we could choose any pair to resolve the singularity. There are three choices, related by triality permutation symmetry. Each such resolution has a complex line of moduli space. These three lines meet at the origin. Therefore, the model has three branches, with a singular point at the origin. This is as predicted in [3].

Perhaps more interestingly, the L-picture suggests that there are further non normalizable deformations, which must be included in the complete moduli space of this model. To see this, note that we may simply translate one of the branes, let us say $\mathrm{D6}_3$, to soften the singularity,

$$D6_3 \ : \quad \frac{1}{2}\vec{\phi}_1 - \frac{\sqrt{3}}{2}\vec{\phi}_2 = \frac{1}{2}\vec{m}\,. \tag{4.54}$$

This deformation breaks the SU(2) rotation symmetry of the IIA background to U(1), so every point with $\vec{m} \neq 0$ corresponds to a different branch. The L-picture is drawn in figure 9. Moreover, we see that if we now resolve $\mathrm{D6}_1$ and $\mathrm{D6}_2$ onto the curve (4.52), then $\mathrm{D6}_3$ does not intersect this curve as long as,

$$|\vec{m}|^2 < \rho\,. \tag{4.55}$$

Note that, unlike the deformation (4.52), the translation of the D6-brane is non-normalizable.

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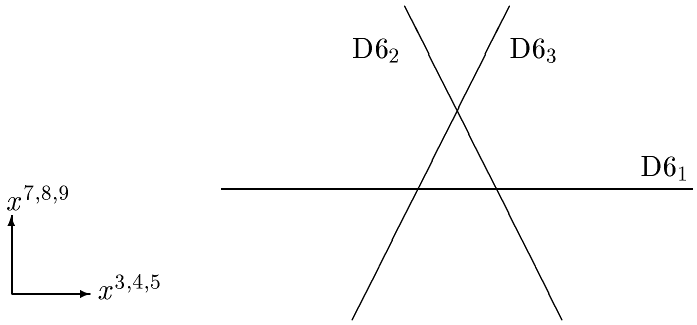

Figure 9: Partially resolved singularity in which the D6$_3$-brane is translated.

Probe theory. Now, let us introduce a probe D2-brane in this background, and look at the $\mathcal{N}=1$ gauge theory on its world-volume. In the singular limit of flat intersecting D6-branes, the addition of the extra D6-brane simply ensures the addition of an extra hypermultiplet,

Theory A: U(1) with 6 scalars and 3 hypermultiplets

As in the case of two intersecting D6-branes, the 6 scalar multiplets combine into two triplets whose interactions with the hypermultiplets are again of the form (4.27),

$$f = \sum_{i=1}^{3}\sum_{a=1}^{2} T_i^a\, W_i^\dagger\, \vec{\tau}\cdot\vec{\Phi}_a W_i \tag{4.56}$$

where the Yukawa matrix is given by,

$$T_i^a = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}. \tag{4.57}$$

The D2-brane world-volume theory enjoys a $\mathrm{U}(1)_J\times \mathrm{SU}(2)\times \mathrm{U}(1)_F^2$ global symmetry. The first three factors act in the same manner as the previous case, while the two U(1) flavor symmetries again act orthogonally to the gauge group; we may take w_i to have charges $(+1,0)$, $(-1,+1)$ and $(0,-1)$ respectively. As the D2-brane world-volume theory flows to the infra-red, we expect the $\mathrm{SU}(2)\times\mathrm{U}(1)$ isometry group of the Coulomb branch to be enhanced to the full SU(3) isometry group of the flag manifold Y.

As discussed above, there are two deformations of the manifold. The first, in which two of the D6-branes combine to form a smooth locus L, induces the same deformation operator (4.33) as in the previous section. The second deformation, in which one of the D6-branes is moved away from the others, is even easier: it corresponds to a triplet of mass parameters for one of the hypermultiplets.

IIB brane construction and the mirror theory. As in the previous section, we may try to determine an algebraic expression for the G_2 manifold, X, by realizing it as the Higgs branch of a mirror theory. Once again, we may bring to bear either the field theoretic or brane construction techniques. We start with the field theory approach. Following the prescription of section 2.1, we find the mirror theory,

Theory B: $\mathrm{U}(1)^2$ with 3 scalar and 3 hypermultiplets

The charges of the three hypermultiplets under the $\mathrm{U}(1)^2$ gauge group are $(+1,-1,0)$ and $(0,+1,-1)$. The three scalar multiplets form a triplet, $\vec{\phi}$, and couple to the hypermultiplets through the real superpotential,

$$f_B = \sum_{i=1}^{3} W_i^{\dagger}\vec{\tau}W_i \cdot \vec{\phi}\,. \tag{4.58}$$

This provides 3 real constraints on the 12 real scalar fields contained in the hypermultiplets. After dividing by the gauge group, we are left with a Higgs branch of dimension 7, as required. It is given by the constraints,

$$\sum_{i=1}^{3} |q_i|^2 - |\tilde{q}_i|^2 = 0\,, \tag{4.59}$$

$$\sum_{i=1}^{3} \tilde{q}_i q_i = 0\,. \tag{4.60}$$

This space has a manifest $\mathrm{SU}(3)$ isometry group, in agreement with our expectations that this is the cone over $\mathrm{SU}(3)/\,\mathrm{U}(1)^2$. Given that the Higgs branch is already a quotient by a $\mathrm{U}(1)^2$ action, it is natural to conjecture that the constraints (4.59), (4.60) taken alone (i.e. before we divide by the gauge group) yield the cone over $\mathrm{SU}(3)$, at least topologically.

In order to see that this is indeed the case, we must find the base of the cone described by the eqs. (4.59), (4.60). To do this, we intersect this space with a sphere of a given radius (it is convenient to choose the radius to be $\sqrt{2}$),

$$\sum_{i=1}^{3} |q_i|^2 + |\tilde{q}_i|^2 = 2\,. \tag{4.61}$$

Then, we can rewrite conditions (4.59) - (4.61) as:

$$\sum_{i=1}^{3} |q_i|^2 = 1\,, \tag{4.62}$$

$$\sum_{i=1}^{3} |\tilde{q}_i|^2 = 1\,, \tag{4.63}$$

$$\sum_{i=1}^{3} \tilde{q}_i q_i = 0\,. \tag{4.64}$$

These equations define a codimension four submanifold in $\mathbb{C}^3 \times \mathbb{C}^3$, parameterized by q_i and $\tilde{q}_i$, the topology of which is to be determined. The first two equations, (4.62) and (4.63), restrict q and $\tilde{q}$ to be unit vectors in $\mathbb{C}^3$, and (4.64) further implies that they are orthogonal. Therefore, the manifold we are looking for can be viewed[5] as a set of orthonormal 2-frames in $\mathbb{C}^3$. By definition, this is a complex Stiefel manifold:

$$V_{3,2}(\mathbb{C}) = \mathrm{U}(3)/\,\mathrm{U}(1) = \mathrm{SU}(3)\,. \tag{4.65}$$

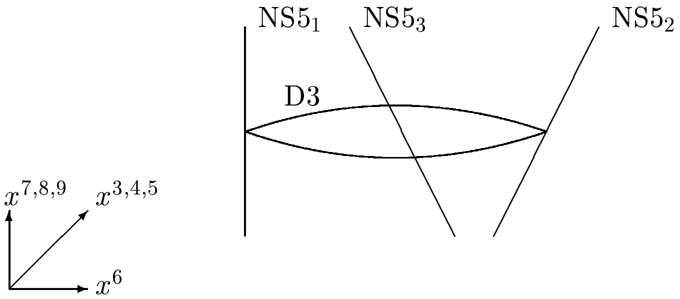

Figure 10: IIB Brane model for Theory B′: U(1)^2 with 3 hypermultiplets.

This proves that the base of the cone defined by the constraints (4.59), (4.60) is isomorphic to SU(3).

The constraints (4.59), (4.60) have a natural deformation which partially resolves the singularity. This is obtained by simply adding constants to the right-hand side of each equation, and corresponds to moving the D6$_3$-brane as described earlier in this section.

One can also attempt to re-derive the mirror theory using brane techniques. As in the previous example, by compactifying the x^6 direction and performing subsequent T- and S-dualities, we can find dual D-brane configurations in type IIB string theory:

$$\begin{array}{rl} D3 & 126 \\ NS5_1 & 12789 \\ NS5_2 & 12[37]_{\theta_1}[48]_{\theta_2}[59]_{\theta_3} \\ NS5_3 & 126[37]_{2\theta_1}[48]_{2\theta_2}[59]_{2\theta_3} \end{array}$$

where

$$\theta_1 = \theta_2 = \theta_3 = \frac{2\pi}{3}\,. \tag{4.66}$$

This brane configuration is drawn in figure 10. Each segment of D3-brane carries a U(1) gauge field, while each intersection yields a hypermultiplet, charged oppositely under the adjacent gauge fields. The overall U(1) decouples, leaving only U(1)^2 gauge group to act on the hypermultiplets. However, there are no further massless, neutral scalar multiplets as in Theory B above. We thus have

Theory B′: U(1)^2 with 3 hypermultiplets

Naively this theory has a Higgs branch of real dimension $12 - 2 = 10$. This clearly cannot be the case. What we have missed is the contribution from the massive neutral scalar fields. As in the case of the generalized conifold [5], these mediate interactions which result in a superpotential for the hypermultiplets. However, as for the Calabi-Yau example, determining the exact superpotential is somewhat more complicated than simply integrating the massive fields at the tree-level. One must follow the anomalous dimension of the various fields under renormalization group flow. This seems out of our reach in the present case,

[5]We thank James Sparks for pointing this out to us.

so instead we conjecture the simplest coupling consistent with all symmetries of the theory which still yields a seven dimensional Higgs branch. It is the quartic superpotential,

$$f'_B = \left(\sum_{i=1}^{3} W_i^\dagger \vec{\tau} W_i - \vec{m} \right)^2 . \tag{4.67}$$

For $\vec{m} = 0$, the Higgs branch of this theory coincides with that of Theory B given in (4.59), (4.60). Non-zero $\vec{m}$ corresponds to moving the $D6_3$-brane. In this fashion, the brane picture yields the same results as the field theory analysis.

4.4 A new model with 3 D6 branes

We now turn to new models, which after reduction to Type IIA theory can be described by three, flat D6-branes lying in the world-volume directions

$$\begin{array}{ll} D6_1 & 123456 \\ D6_2 & 123689 \\ D6_3 & 124679 \end{array}$$

It is easy to check that this configuration of D6-brane is supersymmetric. Indeed, the locus L, which is a union of three special lagrangian 3-planes, is calibrated with respect to

$$\begin{aligned} \mathrm{Re}(\Omega) &= dx^3 \wedge dx^4 \wedge dx^5 + dx^3 \wedge dx^8 \wedge dx^9 + dx^4 \wedge dx^7 \wedge dx^9 + dx^5 \wedge dx^7 \wedge dx^8 \\ &= \mathrm{Re}(dz_1 \wedge dz_2 \wedge dz_3) \end{aligned}$$

and the form $\mathrm{Re}(\Omega)$ restricts to volume form on each 3-plane. This locus L has no continuous isometries although, upon lifting to M-theory the resulting G_2 manifold is expected to have at least a U(1) isometry.

Let us discuss the possible resolutions of the singularity. Notice that, in the absence of any one of the three branes, the remaining two lift to the singular Calabi-Yau conifold discussed in section 3. Ignoring the x^1 and x^2 directions, any pair of branes intersect over a line. All three branes intersect at a point: $x^p = 0$ for $p = 3, \ldots, 9$.

The simplest resolution of the singularity involves separating the branes. There are three parameters, a, b and c, corresponding to the relative separations of the branes which we choose as

$$\begin{array}{lll} D6_1 : x^7 = m_1 \,, & x^8 = 0 \,, & x^9 = 0 \,, \\ D6_2 : x^4 = 0 \,, & x^5 = 0 \,, & x^7 = 0 \,, \\ D6_3 : x^3 = 0 \,, & x^5 = m_2 \,, & x^8 = m_3 \,. \end{array}$$

If all three parameters are non-zero, then the branes do not intersections This is analogous to the small resolution of the conifold singularity. The resulting locus has topology,

$$L = \mathbb{R}^3 \cup \mathbb{R}^3 \cup \mathbb{R}^3 \,. \tag{4.68}$$

From equation (1.4), we see that this IIA background lifts to a manifold X of G_2 holonomy with $h_2(X) = 2$. We will give an algebraic description of this manifold when we come to discuss the IIB brane construction.

There is another way to remove the singularity, which uses the methods of the deformed and the resolved conifold. In this approach, we set $m_a = 0$, and pick two D6-branes, say $D6_1$ and $D6_2$, which intersect over the x^7 line. Defining the complex variables,

$$\psi_1 = x^8 + ix^9 \,, \qquad \psi_2 = x^4 + ix^5 \tag{4.69}$$

the position of the first two D6-branes is described as $\psi_1\psi_2 = 0$ and $x^7 = 0$. We now deform the first of these equations to

$$\psi_1\psi_2 = \rho \equiv \rho_1 + i\rho_2 \,. \tag{4.70}$$

Together with the equation for the flat D6$_3$-brane, this defines a special lagrangian curve. To see this, it suffices to note that D6-branes lying on the curve (4.70) preserve the same supersymmetry as the flat, intersecting branes. The D6$_3$-brane intersects the complex curve (4.70) only asymptotically, ensuring that the singularity is indeed removed. The smooth locus of D6-branes has topology

$$L = S^1 \times \mathbb{R}^2 \cup \mathbb{R}^3 \,. \tag{4.71}$$

Once more employing equations (1.4), we see that this D6-brane configuration lifts to a manifold X of G_2-holonomy with Betti numbers:

$$h_2(X) = 1 \,, \qquad h_3(X) = 1 \,.$$

This manifold therefore admits a geometric transition. From the L-picture, it is clear that this is entirely analogous to the conifold transition, in which an $\mathbf{S}^2$ shrinks, and an $\mathbf{S}^3$ grows. The L-picture suggests that, as for the conifold, this process necessarily involves passing through a singular point. Also, since each deformation is equivalent to that in the conifold, they are non-normalizable.

Finally we note that, with three D6-branes, we may perform both deformation and resolution simultaneously. This involves deforming, say, the D6$_1$ and D6$_2$-brane as in (4.70), while simply moving the D6$_3$-brane in the $(x^5 = m_2) - (x^8 = m_3)$-plane. If the D6$_3$-brane moves a short distance away from the origin, it will intersect the complex curve (4.70), resulting in a singular manifold. However, if we move them into the region defined by,

$$m_2^2 m_3^2 - m_2 m_3 \rho_2 > \frac{1}{4}\rho_1^2 \tag{4.72}$$

then the G_2 lift becomes smooth once again. It is interesting to note that this condition is opposite to the corresponding condition (4.55) for the cone over $\mathrm{SU}(3)/\mathrm{U}(1)^2$ discussed in the previous section. In the latter case, deforming two of the D6-branes created a hole through which the third could pass. In the present case, however, deforming two of the D6-branes does not create a hole. Rather, the third D6-brane must be moved sufficiently far from the initial two in order to avoid them. (Alternatively, as mentioned above, if it is left at the origin, it intersects only asymptotically.)

The various phases of the manifold X we just discussed are listed in table 3. The dots in the last row indicate that our analysis is by no means complete, and there might be more phases to be discovered.

Betti Numbers	Number of Phases	Number of Deformations
$h_2 = 2,\, h_3 = 0$	1	3
$h_2 = 1,\, h_3 = 1$	3	2
$h_2 = 1,\, h_3 = 1$	3	4
...	...	...

Table 3: Some topological phases of a manifold X with G_2 holonomy.

Probe theory. As usual, we probe this brane set-up with a D2-brane extended in the $1-2$ plane. As in previous sections, the probe theory has $\mathcal{N}=1$ supersymmetry, and is simple to write down,

Theory A: U(1) with 6 scalar and 3 hypermultiplets

If ϕ_a, $a=1,\ldots,6$, denote the fields corresponding to D2-brane motion along $x^{3,4,5,7,8,9}$ then we have the real superpotential,

$$f = \sum_{i=1}^{3} \vec{A}_i \cdot W_i^{\dagger} \vec{\tau} W_i\,, \tag{4.73}$$

where the triplets $\vec{A}_i$ are suitable combinations of the Φ_i,

$$\vec{A}_1 = (\Phi_7, \Phi_8, \Phi_9)\,, \qquad \vec{A}_2 = (\Phi_7, \Phi_4, \Phi_5)\,, \qquad \vec{A}_3 = (\Phi_3, \Phi_8, \Phi_5)\,. \tag{4.74}$$

The deformations of this model are the same as those described in the section on the conifold. The deformation (4.70) is given by the addition of the superpotential (3.9). The translations of the D6-branes correspond to mass parameters. For example, if we set $\rho=0$ in (4.70), but include the translation parameters m_a, then the scalar potential is,

$$\begin{aligned} V_A = {} & e^2\left(\mathrm{Re}(\tilde{q}_1 q_1) + \mathrm{Re}(\tilde{q}_2 q_2)\right)^2 + e^2\,\mathrm{Re}(\tilde{q}_3 q_3)^2 + \left((\phi_7 - m_1)^2 + \phi_8^2 + \phi_9^2\right) w_1^{\dagger} w_1 + \\ & + e^2\left(\mathrm{Im}(\tilde{q}_1 q_1) + \mathrm{Im}(\tilde{q}_3 q_3)\right)^2 + e^2\,\mathrm{Im}(\tilde{q}_2 q_2)^2 + \left(\phi_4^2 + \phi_5^2 + \phi_7^2\right) w_2^{\dagger} w_2 + \\ & + e^2\left(|q_1|^2 - |\tilde{q}_1|^2\right)^2 + e^2\left(|q_2|^2 - |\tilde{q}_2|^2 + |q_3|^2 - |\tilde{q}_3|^2\right)^2 + \\ & + \left((\phi_3^2 + (\phi_5 - m_2)^2 + (\phi_8 - m_3)^2\right) w_3^{\dagger} w_3\,. \end{aligned}$$

Upon the lift to M-theory, this configuration of D6-branes yields a G_2-manifold X with a single U(1) isometry that comes from $\mathrm{U}(1)_J$ symmetry and $\mathrm{U}(1)^2$ flavor symmetry that is inherited from gauge symmetries on D6-branes. As usual, the diagonal U(1) decouples. Hence we expect that when X develops a conical singularity the total global symmetry group is:

$$K_Y = \mathrm{U}(1)_J \times \mathrm{U}(1)_C^2\,. \tag{4.75}$$

We will return to the geometry of spaces X and Y in a moment.

IIB brane construction and the mirror theory. In the previous section we have determined the topology of the manifold X. Here we shall give an algebraic description of the manifold using mirror symmetry. As we have many times, we perform a T-duality along the x^6 direction, followed by an S-duality. We end up with a IIB brane construction with three NS5-branes, and a D3-brane wrapped on the compact x^6 direction:

$$\begin{aligned} NS5_1 &: \ 12345 \\ NS5_2 &: \ 12389 \\ NS5_3 &: \ 12479 \\ D3 &: \ 126 \end{aligned}$$

At this stage, the reader with a (very) good memory will recognize this as the model considered as Example 2 in section 2.2, where we derived the mirror theory to be

Theory B: $\mathrm{U}(1)^2$ with 3 scalar and 3 hypermultiplets

where we have ignored the overall, free U(1) gauge symmetry. The hypermultiplets may be taken to have charges $(+1,-1,0)$ and $(0,-1,+1)$ under the gauge group. In section 2.2 we further found both Yukawa and quartic superpotential interactions, given in equations (2.22) and (2.24) respectively. These yield the constraints,

$$\begin{aligned} |q_1|^2 - |\tilde{q}_1|^2 - |q_2|^2 + |\tilde{q}_2|^2 &= 0\,, \\ \mathrm{Re}(\tilde{q}_2 q_2 - \tilde{q}_3 q_3) &= 0\,, \\ \mathrm{Im}(\tilde{q}_3 q_3 - \tilde{q}_1 q_1) &= 0\,. \end{aligned}$$

These give 3 real constraints on the 12 real parameters w_i. After dividing by the $\mathrm{U}(1)^2$ gauge action, we arrive at the Higgs branch. Since the above set of equations is invariant under rescaling of the fields q, it describes a conical space X.

As we have seen, the locus L has a rich moduli space of deformations. The deformations which blow up a $\mathbf{S}^3$, given by (4.70) are realized by the mirror operator (3.12), relevant to the deformed conifold. This is poorly understood. More simple are the translations of the D6-branes which blow up $\mathbf{S}^2$'s. From the brane picture, we see that these correspond to FI terms. Specifically, we have the additional real superpotential,

$$\Delta f = \sum_{a=1}^{3} m_a \Phi_a\,. \tag{4.76}$$

However, this cannot be the full answer because this interaction partially lifts the Higgs branch, setting $W_i^\dagger \tau^i W_i = 0$ for $i = 1, 2, 3$. In order to compensate for this, we must further add the bare mass terms,

$$\Delta f = \sum_{i=1}^{3} m_i W_i^\dagger \tau^i W_i\,. \tag{4.77}$$

The theory once again has a seven dimensional Higgs branch, now given by the constraints,

$$\begin{aligned} |q_1|^2 - |\tilde{q}_1|^2 - |q_2|^2 + |\tilde{q}_2|^2 &= m_1\,, \\ \mathrm{Re}(\tilde{q}_2 q_2 - \tilde{q}_3 q_3) &= m_2\,, \\ \mathrm{Im}(\tilde{q}_3 q_3 - \tilde{q}_1 q_1) &= m_3 \end{aligned}$$

modulo the $\mathrm{U}(1)^2$ gauge action.

Finally, we comment that the above Higgs branch description of X may also be recovered using the field theory method of describing $\mathcal{N} = 1$ duals presented in section 2.1. In this case, one finds the mirror theory to have the same field content as Theory B, but without the quartic superpotential (2.24). However, as discussed at length in section 2.2, the quartic superpotential does not add any further constraints to determine the Higgs branch, and the two techniques therefore agree.

Generalizations. This model may be extended to consist of many, mutually orthogonal D6-branes:

$$\begin{array}{lll} N_1 & D6_1 & 123456 \\ N_2 & D6_2 & 123689 \\ N_3 & D6_3 & 124679 \end{array}$$

The moduli space of deformations is a simple generalization of that considered above, where the singularity appearing on the intersection of any pair of D6-branes may be resolved either through translation or complex deformation. At least part of the physics associated with compactification of M-theory on such a G_2 manifold X is obvious: as X develops a conical singularity, non-abelian degrees of freedom appear with gauge group $\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \mathrm{U}(N_3)$. It would be interesting to understand the resulting G_2 manifolds further – they seem to be analogous to the generalized conifold studied, for example, in [55]. It would also be interesting to see if these manifolds can be obtained from partial resolution of the orbifold $X = \mathbb{R}^7/\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \mathbb{Z}_{k_3}$, where the orbifold group is generated by three elements α, β, and γ of order k_1, k_2, and k_3, respectively:

$$\begin{array}{ll} \alpha : (x^7 + ix^8) \mapsto e^{2\pi i/k_1}(x^7 + ix^8), & (x^9 + ix^{11}) \mapsto e^{-2\pi i/k_1}(x^9 + ix^{11}) \\ \beta : (x^4 + ix^5) \mapsto e^{2\pi i/k_2}(x^4 + ix^5), & (x^7 + ix^{11}) \mapsto e^{-2\pi i/k_2}(x^7 + ix^{11}) \\ \gamma : (x^3 + ix^5) \mapsto e^{2\pi i/k_3}(x^3 + ix^5), & (x^8 + ix^{11}) \mapsto e^{-2\pi i/k_3}(x^8 + ix^{11}) \end{array}$$

An example of such space, with $k_i = 2$, was recently studied in [17]. Since X is orbifold, the base of the cone is simply $Y = \mathbf{S}^6/\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \mathbb{Z}_{k_3}$. Note, however, that orbifold action has fixed points (of codimension 4) on $\mathbf{S}^6$.

4.5 A new model with 4 D6 branes

A careful reader might notice that one can add an additional D6-brane to the configuration of 3 orthogonally intersection D6-branes, discussed in the previous section:

$$\begin{array}{ll} D6_1 & 123456 \\ D6_2 & 123689 \\ D6_3 & 124679 \\ D6_4 & 125678 \end{array}$$

The $\mathrm{D6}_4$ brane does not break supersymmetry further since the special lagrangian calibration $\mathrm{Re}(\Omega)$ restricts to the volume form on the $x^{5,7,8}$-plane. All of the four D6-branes are on an equal footing.

As in the previous section, we may resolve the singularity using the two deformations of the conifold. Firstly, let us consider translating the D6-branes. There are six such parameters,

$$\begin{array}{lll} D6_1 : x^7 = m_1\,, & x^8 = 0\,, & x^9 = 0 \\ D6_2 : x^4 = 0\,, & x^5 = 0\,, & x^7 = 0 \\ D6_3 : x^3 = 0\,, & x^5 = m_2\,, & x^8 = m_3 \\ D6_4 : x^3 = n_1\,, & x^4 = n_2\,, & x^9 = n_3 \end{array}$$

If $m_i, n_i \neq 0$ for all $i = 1, 2, 3$, then the D6-branes do not intersect and locus L has topology,

$$L = \mathbb{R}^3 \cup \mathbb{R}^3 \cup \mathbb{R}^3 \cup \mathbb{R}^3 \tag{4.78}$$

so, from (1.4), we see that the lift to M-theory results in a smooth manifold X of G_2 holonomy with topology $h_2(X) = 3$, with no other non-trivial cycles. As we shall now show, there are geometrical transitions which involve blowing down a maximum of two of these $\mathbf{S}^2$'s and replacing them with $\mathbf{S}^3$'s. There are therefore three topologically distinct manifolds.

Let us first consider blowing up a single $\mathbf{S}^3$. This requires us to pick a pair of D6-branes, and there are therefore six independent ways of doing this. Here we give an example. Let us define the two complex variables:

$$\begin{array}{ll} \psi_1 = x^8 + ix^9\,, & \psi_2 = x^4 + ix^5 \\ \tilde{\psi}_1 = x^5 + ix^8\,, & \tilde{\psi}_2 = x^4 + ix^9 \end{array}$$

If we set $m_i = n_2 = n_3 = 0$, then we can deform the first two D6-branes into the complex curve $\psi_1\psi_2 = \rho$, in which case the singularity is removed providing $n_1 \neq 0$. As in the previous example, we may move the $D6_3$ and $D6_4$ brane by turning on m_2, m_3 and n_2, n_3 respectively. The lift to M-theory is again non-singular providing we move them far enough. Alternatively, we may set $n_i = m_2 = m_3 = 0$, and then deform the $D6_3$ and $D6_4$ branes on the curve $\tilde{\psi}_1\tilde{\psi}_2 = \tilde{\rho}$. Again, the special lagrangian manifold is smooth if $m_1 \neq 0$. In each of these cases, the locus L has topology:

$$L = S^1 \times \mathbb{R}^2 \cup \mathbb{R}^3 \cup \mathbb{R}^3 \tag{4.79}$$

which ensures that the lift to M-theory results in a manifold X with $h_2(X) = 2$, and $h_3(X) = 1$.

Finally, we may resolve two pairs of D6-branes. There are three ways of choosing two pairs. Let us examine the deformation of the $D6_1$–$D6_2$ pair, and the $D6_3$–$D6_4$ pair:

$$\psi_1\psi_2 = \rho, \qquad \tilde{\psi}_1\tilde{\psi}_2 = \tilde{\rho}\,. \tag{4.80}$$

For suitable choices of the parameters, the two curves do not intersect (e.g. $\mathrm{Re}\,\rho = \mathrm{Im}\,\tilde{\rho} = 0$). The locus L has topology

$$L = (S^1 \times \mathbb{R}^2) \cup (S^1 \times \mathbb{R}^2) \tag{4.81}$$

so the smooth manifold X has Betti numbers:

$$h_2(X) = 1\,, \qquad h_3(X) = 2\,.$$

As in the previous example, since each of these deformations is inherited from the conifold, they are non-normalizable.

Probe theory. The $\mathcal{N}=1$ theory living on a probe D2-brane is again easy to determine:

Theory A: U(1) with 6 scalar and 4 hypermultiplets

with a superpotential given by:

$$f = \sum_{i=1}^{4} \vec{A}_i \cdot W_i^\dagger \vec{\tau} W_i \,, \tag{4.82}$$

where the triplets $\vec{A}_i$ where given in equation (4.74) for $i = 1, 2, 3$, and the fourth is,

$$\vec{A}_4 = (\Phi_3, \Phi_4, \Phi_9) \,. \tag{4.83}$$

As before, we write the scalar potential only for the model with $h_2(X) = 3$, which consists of four, separated flat D6-branes:

$$\begin{aligned} V_A = {} & e^2 \left(\mathrm{Re}(\tilde{q}_1 q_1) + \mathrm{Re}(\tilde{q}_2 q_2)\right)^2 + e^2 \left(\mathrm{Re}(\tilde{q}_3 q_3) + \mathrm{Re}(\tilde{q}_4 q_4)\right)^2 + \\ & + e^2 \left(\mathrm{Im}(\tilde{q}_1 q_1) + \mathrm{Im}(\tilde{q}_3 q_3)\right)^2 + e^2 \left(\mathrm{Im}(\tilde{q}_2 q_2)^2 + \mathrm{Im}(\tilde{q}_4 q_4)\right)^2 + \\ & + e^2 \left(|q_1|^2 - |\tilde{q}_1|^2 + |q_4|^2 - |\tilde{q}_4|^2\right)^2 + e^2 \left(|q_2|^2 - |\tilde{q}_2|^2 + |q_3|^2 - |\tilde{q}_3|^2\right)^2 + \\ & + \left((\phi_7 - m_1)^2 + \phi_8^2 + \phi_9^2\right) w_1^\dagger w_1 + \left(\phi_4^2 + \phi_5^2 + \phi_7^2\right) w_2^\dagger w_2 + \\ & + \left((\phi_3^2 + (\phi_5 - m_2)^2 + (\phi_8 - m_3)^2\right) w_3^\dagger w_3 + \\ & + \left((\phi_3 - n_1)^2 + (\phi_4 - n_2)^2 + (\phi_9 - n_3)^2\right) w_4^\dagger w_4 \,. \end{aligned}$$

Upon the lift to M-theory, the manifold X has a single U(1) isometry arising from $\mathrm{U}(1)_J$ symmetry. In the singular conical case, there is a further $\mathrm{U}(1)^3_C$ that comes from the non-diagonal U(1) gauge symmetries on D6-branes.

$$K_Y = \mathrm{U}(1)_J \times \mathrm{U}(1)^3_C \,. \tag{4.84}$$

IIB brane construction and the mirror theory. As in the previous section, we shall endeavor to find an algebraic expression for X Upon compactifying the x^6 direction, and performing both T- and S-dualities, we end up with a Hanany-Witten type periodic model, consisting of:

$$\begin{aligned} NS5_1: &\ \ 12345 \\ NS5_2: &\ \ 12389 \\ NS5_3: &\ \ 12479 \\ NS5_4: &\ \ 12578 \\ D3: &\ \ 126 \end{aligned}$$

It has a low-energy $\mathcal{N}=1$ three-dimensional description given by:

Theory B: $\mathrm{U}(1)^3$ with 4 scalar and 4 hypermultiplets where we have ignored the overall, free U(1) gauge symmetry. The hypermultiplets may be taken to have charges $(+1,-1,0,0)$, $(0,+1,-1,0)$, and $(0,0,+1,-1)$ under the three gauge groups. The Yukawa couplings are simple to write down following the prescription given in the previous section

$$\begin{aligned} f_{\text{Yuk}} =& \Phi_1 \left(W_1^\dagger\tau^3 W_1 - W_2^\dagger\tau^3 W_2\right) + \Phi_2 \left(W_2^\dagger\tau^1 W_2 - W_3^\dagger\tau^1 W_3\right) + \\ &+ \Phi_3 \left(W_3^\dagger\tau^3 W_3 - W_4^\dagger\tau^3 W_4\right) + \Phi_4 \left(W_4^\dagger\tau^1 W_4 - W_1^\dagger\tau^1 W_1\right), \end{aligned} \tag{4.85}$$

where we have taken the liberty of performing a field redefinition of W_3 relative to the previous section to make the symmetries more manifest. Notice that this time there is an important difference from the three D6-brane case discussed in the previous section. In that case, the Yukawa terms alone were enough to result in a Higgs branch of real dimension seven; any quartic superpotential was required to be of a form that didn't impose any further constraints, which indeed was what we found. In the present case with four D6-branes however, this is no longer the case. The Yukawa couplings above, together with the $\mathrm{U}(1)^3$ gauge action, result in a Higgs branch of real dimension nine. We therefore expect the quartic superpotential to yield two further real constraints. This is an important check on the answer below.

The quartic superpotential can be easily written down following the discussion of section 2.2, and [5]:

$$\begin{aligned} f_4 =& \left(W_1^\dagger\tau^1 W_1\, W_2^\dagger\tau^1 W_2 - W_1^\dagger\tau^2 W_1\, W_2^\dagger\tau^2 W_2\right) + \\ &+ \left(W_2^\dagger\tau^2 W_2\, W_3^\dagger\tau^2 W_3 - W_2^\dagger\tau^3 W_2\, W_3^\dagger\tau^3 W_3\right) - \\ &- \left(W_3^\dagger\tau^1 W_3\, W_4^\dagger\tau^1 W_4 - W_3^\dagger\tau^2 W_3\, W_4^\dagger\tau^2 W_4\right) - \\ &- \left(W_4^\dagger\tau^2 W_4\, W_1^\dagger\tau^2 W_1 - W_1^\dagger\tau^3 W_1\, W_4^\dagger\tau^3 W_4\right). \end{aligned} \tag{4.86}$$

Reassuringly, when combined with the Yukawa couplings above, this does indeed lead to 6 real constraints on the 16 hypermultiplet degrees of freedom,

$$\begin{aligned} |q_1|^2 - |\tilde{q}_1|^2 - |q_2|^2 + |\tilde{q}_2|^2 &= 0 \\ |q_3|^2 - |\tilde{q}_3|^2 - |q_4|^2 + |\tilde{q}_4|^2 &= 0 \\ \mathrm{Re}(\tilde{q}_2 q_2 - \tilde{q}_3 q_3) &= 0 \\ \mathrm{Re}(\tilde{q}_4 q_4 - \tilde{q}_1 q_1) &= 0 \\ \mathrm{Im}(\tilde{q}_1 q_1 - \tilde{q}_3 q_3) &= 0 \\ \mathrm{Im}(\tilde{q}_2 q_2 + \tilde{q}_4 q_4) &= 0 \end{aligned} \tag{4.87}$$

where the first four constraints arise from the Yukawa couplings alone, and the last two come from the quartic superpotential. After dividing by the $\mathrm{U}(1)^3$ gauge action, we arrive at the seven dimensional Higgs branch. Once again, the above set of equations is invariant under rescaling of the fields q, and so describes a conical singularity X.

We now turn to the deformations corresponding to separating the D6-branes. As before, we expect these to correspond to FI parameters. In fact, only four of them are

FI parameters: m_1, m_2, n_1 and n_3 each correspond to separating adjacent NS5-branes, and so give rise to FI parameters. As in the previous section, it is necessary that these are accompanied by bare mass parameters for hypermultiplets so as not to lift the Higgs branch. In contrast, m_3 and n_2 correspond to separating opposite NS5-branes. These induce only the mass terms for the hypermultiplets,

$$\Delta f = m_3 \left(W_2^\dagger \tau^2 W_2 + W_4^\dagger \tau^2 W_4 \right) + n_2 \left(W_3^\dagger \tau^2 W_3 - W_1^\dagger \tau^2 W_1 \right). \tag{4.88}$$

The net result of these deformations is to add constant terms to the right-hand-side of (4.87),

$$\begin{aligned} |q_1|^2 - |\tilde{q}_1|^2 - |q_2|^2 + |\tilde{q}_2|^2 &= m_1 \\ |q_3|^2 - |\tilde{q}_3|^2 - |q_4|^2 + |\tilde{q}_4|^2 &= n_1 \\ \mathrm{Re}(\tilde{q}_2 q_2 - \tilde{q}_3 q_3) &= m_2 \\ \mathrm{Re}(\tilde{q}_4 q_4 - \tilde{q}_1 q_1) &= n_3 \\ \mathrm{Im}(\tilde{q}_1 q_1 - \tilde{q}_3 q_3) &= m_3 \\ \mathrm{Im}(\tilde{q}_2 q_2 + \tilde{q}_4 q_4) &= n_2 \end{aligned} \tag{4.89}$$

Again, this model is but the simplest example of a family of models consisting of N_i of the D6$_i$-brane. It would be of interest to examine if these too can be thought of as partial resolutions of a $\mathbb{R}^7$ orbifold.

Finally, let us mention that we can also derive the same result for the Higgs branch using the field theory techniques explained in section 2.1. In this case, however, the mirror theory differs somewhat from that obtained using brane techniques. Specifically, we find:

Theory B′: $\mathrm{U}(1)^3$ with 6 scalar and 4 hypermultiplets

The gauge multiplets and the first four hypermultiplets have the same couplings as in Theory B above. However, Theory B′ has only Yukawa couplings, and no quartic superpotential. The two extra scalars in Theory B′ play the role of the quartic superpotential by imposing the two extra constraints (the imaginary equations in (4.89)). The two methods therefore yield the same Higgs branch and, indeed, the same physics at scales below those set by the mass terms m_3 and n_2.

4.6 The cone over $\mathbf{S}^3 \times \mathbf{S}^3$

Let us now turn to the most subtle example discussed in [3], that of a cone over $Y = \mathbf{S}^3 \times \mathbf{S}^3$. This has proven to be an extremely interesting manifold, providing a new handle on four dimensional $\mathcal{N} = 1$ super Yang-Mills [60, 25, 3]. Algebraically this dimension 7 space X is described by the simple equation:

$$\sum_{i=1}^{2} |q_i|^2 - |\tilde{q}_i|^2 = m \tag{4.90}$$

for real m, and is topologically an $\mathbb{R}^4$ bundle over $\mathbf{S}^3$. This space has at least four interesting reductions to IIA string theory. As well as the two discussed above — namely, reduction on a transverse circle, and the L-picture — we could also choose to embed the M-theory circle

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either in the $\mathbb{R}^4$ fiber, or the $\mathbf{S}^3$ base. These reductions result in the deformed conifold with a wrapped D6-brane or the resolved conifold with RR flux, respectively [60, 25, 3]. D2-brane probes on the IIA background with flux were previously discussed by Aganagic and Vafa [34]. Their method also yields the theory on the probe of X and, as we shall show, mirror symmetry gives the probe of the L-picture. The theory on a D2-brane probe of a D6-brane wrapping the deformed conifold remains elusive.

Let us start by reducing on an $\mathbf{S}^1 \subset \mathbf{S}^3$. The resulting IIA conifiguration is the resolved conifold with a single unit of RR two-form flux through the two-cycle. The theory on a transverse D2-brane probe of the resolved conifold was reviewed in section 3. It has $\mathcal{N}=2$ supersymmetry and a Higgs branch, which reproduces the geometry of the conifold. Recall that it comes complete with a free $\mathcal{N}=2$ vector multiplet, including a gauge field $\tilde{A}$ corresponding to the motion of D2-brane along the M-theory circle, as well as a scalar describing the direction transverse to the conifold. The addition of the RR flux through the two-cycle breaks supersymmetry on the D2-brane to $\mathcal{N}=1$ by inducing a CS-coupling, $\tilde{A} \wedge F$, between the two $\mathcal{N}=1$ vector multiplets of the theory [34]. The simplest way to see this is to examine the Wess-Zumino terms on a fractional D4-brane wrapped around the $\mathbf{S}^2$. The resulting $\mathcal{N}=1$ theory is,

Theory A: $\mathrm{U}(1) \times \tilde{\mathrm{U}(1)}$ with 1 real scalar and 2 hypermultiplets

with a further, free scalar multiplet which we ignore. The hypermultiplets are charged only under the U(1) gauge group, with the $\tilde{\mathrm{U}(1)}$ coupling only through the CS term,

$$\frac{1}{e^2}F^2 + \frac{1}{\tilde{e}^2}\tilde{F}^2 + A \wedge \tilde{F}\,. \tag{4.91}$$

Upon exchanging $\tilde{A}$ for its dual photon $\tilde{\sigma}$, we may replace the above interaction with

$$\frac{1}{e^2}F^2 + \tilde{e}^2\left(\partial\tilde{\sigma} + A\right)^2 \tag{4.92}$$

so that $\tilde{\sigma}$ transforms transitively under the U(1) gauge action. The scalar potential is the same as for the conifold (3.4)

$$V_A = e^2\left(|q_1|^2 + |q_2|^2 - |\tilde{q}_1|^2 - |\tilde{q}_2|^2 - m\right)^2 + \phi^2\left(w_1^\dagger w_1 + w_2^\dagger w_2\right) \tag{4.93}$$

whose Higgs branch reproduces the resolved conifold $O(-1) + O(-1) \to \mathbb{CP}^1$. The transformation of the dual photon $\tilde{\sigma}$ ensures it has unit Hopf fibration over the zero section $\mathbf{P}^1$, yielding a space of the requisite topology [34]. To see this, note that the asymptotic radius of the M-theory circle is $\tilde{e}^2$. In the strong coupling limit $\tilde{e}^2 \to \infty$, the dual photon removes the U(1) gauge invariance, and the effect of the CS coupling is simply to ungauge the original U(1) . We are then left with:

Theory A$'$: 1 scalar multiplet and 2 hypermultiplets

which has the scalar potential (4.93), but no gauge fields. The vacuum moduli space coincides with the equation (4.90) for X.

Both Theories A and A$'$ acquire a new branch of vacua as the three-cycle collapses, $m \to 0$. For the conifold, the interpretation is clear: this branch describes fractional D2 branes, corresponding to D4-anti-D4 branes wrapping the collapsed 2-cycle. Lifting this picture to M-theory, and including the twist of the dual-photon, the interpretation is equally clear in the G_2 case: the fractional D2-branes correspond to M5-anti-M5 branes wrapping the collapsed three-cycle.

In [34], it was further argued that including the Chern-Simons term $kA \wedge \tilde{F}$ results in the $\mathbb{Z}_k$ quotient of X with topology $\mathbf{S}^3/\mathbb{Z}_k \times \mathbb{R}^4$. In this case, we see that the U(1) action on $\tilde{\sigma}$ leaves a remanant $\mathbb{Z}_k$. The resulting vacuum moduli space is therefore (4.90), with the identification,

$$q_i \to \omega q_i \,, \qquad \tilde{q}_i \to -\omega \tilde{q}_i \tag{4.94}$$

where $\omega^k = 1$. This theory therefore describes the space $(S^3 \times \mathbb{R}^4)/\mathbb{Z}_k$, which is isomophic to $\mathbf{S}^3/\mathbb{Z}_k \times \mathbb{R}^4$ in agreement with [34].

Let us return to the $k = 1$ case, and reduce to IIA on a circle transverse to both X and the M2-brane. The resulting probe theory remains Theory A$'$ above, now with a further free $\mathcal{N} = 1$ vector multiplet describing the motion of the D2-brane in the M-theory direction. In other words, this theory describes IIA string theory on X: we have travelled clockwise around the circle of dualities in figure 1.

Our final type IIA dual can be obtained by embedding M-theory circle in the $\mathbf{S}^3$ inside X, such that [3]:

$$X/\mathbf{S}^1 \cong \mathbb{R}^6 \,.$$

Unlike the other two cases, here the circle has fixed points on a codimention four locus $L \subset \mathbb{R}^6$, which gets identified with the position of D6-branes. Let us determine what L looks like. From (1.4), we see that it has topology:

$$L \cong \mathbf{S}^1 \times \mathbb{R}^2 \,. \tag{4.95}$$

More explicitly, if we parameterise $\mathbb{R}^6 \cong \mathbb{C}^3$ by complex coordinates z_i, $i = 1, 2, 3$, we can write L as [61, 62]:

$$L = \left\{ |z_1|^2 - m^2 = |z_2|^2 = |z_3|^2, \mathrm{Im}(z_1 z_2 z_3) = 0, \mathrm{Re}(z_1 z_2 z_3) \geq 0 \right\} . \tag{4.96}$$

It is easy to check that it indeed has the right topology. In order to see that L is (special) lagrangian, it is convenient to rewrite the Kähler form as:

$$J = \sum_{i=1}^{3} d|z_i|^2 \wedge d\theta_i \,.$$

The defining equations for L can be expressed in the form:

$$\begin{aligned} |z_1|^2 - |z_2|^2 &= m^2 \,, \\ |z_2|^2 - |z_3|^2 &= 0 \,, \\ \theta_1 + \theta_2 + \theta_3 &= 0 \,. \end{aligned} \tag{4.97}$$

Since all of these relations are linear, it is straighforward to check that J restricts to zero on L. A little bit more work shows that L is *special* lagrangian. Furthermore, in the limit $t \to 0$ it degenerates into a cone over $\mathbf{S}^1 \times \mathbf{S}^1$:

$$L_0 = \{|z_1| = |z_2| = |z_3|, \mathrm{Im}(z_1 z_2 z_3) = 0, \mathrm{Re}(z_1 z_2 z_3) \geq 0\}\,. \tag{4.98}$$

Our next task is to reproduce L as the locus of massless hypermultiplets in the mirror theory. Using the standard mirror symmetry techniques, the mirror $\mathcal{N} = 1$ theory is:

Theory B: $\mathrm{U}(1)^2$ with 5 scalar and 2 hypermultiplets

The hypermultiplets have charge $(+1, -1)$ under the first $\mathrm{U}(1)$, and charge $(+1, +1)$ under the second. If we were to neglect one of the $\mathrm{U}(1)$ vector multiplets, this would be precisely the matter content which realises the resolved conifold on the Coulomb branch. Indeed, the coupling of the scalars agrees with that of the conifold theory. In the notation of a real superpotential, we have:

$$f = \sum_{i=1}^{2} \vec{A}_i \cdot W_i^\dagger \vec{\tau} W_i \tag{4.99}$$

where the triplets of scalar multiplets are:

$$\vec{A}_1 = (\Phi_1, \Phi_2, \Phi_3)\,, \qquad \vec{A}_2 = (\Phi_1 - m, \Phi_4, \Phi_5) \tag{4.100}$$

which results in the scalar potential (3.3). Note that the size of the $\mathbf{S}^3$ no plays the role of a mass parameter.

The Coulomb branch of this theory is described by the two dual photons and the real scalar, together with the two chiral multiplets, completing a seven real dimensional manifold. To determine the locus L, we must first decide which of the $\mathrm{U}(1)$ factors to nominate as the M-theory circle. For simplicity, we choose the diagonal combination. Only the first hypermultiplet is charged under this, and becomes massless at $\phi_i = 0$ for $i = 1, 2, 3$. At one-loop level, the dual photon degenerates at these points, leaving a locus of fixed points parameterised by ϕ_4, ϕ_5 and the linear combination of dual photons, $\sigma_1 - \sigma_2$. This latter scalar has non-vanishing period on L as long as $m \neq 0$. This ensures that the locus L does indeed have topology $\mathbf{S}^1 \times \mathbb{R}^2$ as required.

Although Theory B correctly reproduces the topology of L, several puzzles remain. Firstly, it is unclear why a D-brane probe in flat space would have two $\mathrm{U}(1)$ gauge fields on its world-volume, as suggested by the above analysis. Secondly, we have been unable to reproduce the famous three-legged topology for the moduli space of X from the probe theory. This remains an open problem.

5. Sp(2) holonomy

Starting with this section, we turn to compactifications of M-theory on eight dimensional manifolds X of special holonomy. We shall start with a discussion of hyperkähler manifolds with Sp(2) holonomy group since here we have much more control than the Spin(7) models to be discussed in the following section. D2-brane probes of hyperKähler 8-folds have $\mathcal{N} = 3$ supersymmetry on their worldvolume.

There are several features that distinguish M-theory on manifolds X of dimension eight (or greater). For example, suppose that X is a non-compact manifold of dimension eight (at this point we make no specific assumptions about holonomy of X). Suppose, further, that X develops an isolated conical singularity. Motivated by the analysis in the previous sections, one might be interested in a dual type IIA description that involves intersecting D6-branes in topologically flat space-time. If it exists, such a description can be obtained from M-theory via reduction on a circle, such that:

$$X/\,\mathrm{U}(1) \cong \mathbb{R}^7\,. \tag{5.1}$$

The fixed point set, L, of the circle action has codimension four and can be understood as a D6-brane locus [3, 4]. It turns out, that if X develops an isolated conical singularity, L is also conical. What is peculiar about compactification on eight-manifolds is that L can not be represented by intersection of two coassociative planes in $\mathbb{R}^7$; generically, two 4-planes in a 7-dimensional space intersect over a set of dimension one. In this sense, models with X of dimension eight are hard.

Another problem with compactifications on 8-manifolds is that we no longer have extra spatial dimension to perform the M-theory flip. Therefore, there is no string theory method to derive a mirror three-dimensional gauge theory, and we must resort to the field theory techniques of section 2.

Keeping these peculiar subtleties in mind, let's begin with the simplest example of a hyperkähler manifold X given by cotangent bundle of $\mathbb{CP}^2$:

$$X = T^*\mathbb{CP}^2\,.$$

M-theory on this manifold has very interesting dynamics. For example, a membrane anomaly requires the introduction of non-zero G-flux in order to obey the shifted quantization condition:

$$\int_{\mathbb{CP}^2} \frac{G}{2\pi} \in \mathbb{Z} + \frac{1}{2}\,.$$

This flux generates a Chern-Simons coupling in the effective three-dimensional theory, thus breaking the parity symmetry [4, 63]. Its value becomes an additional parameter, and for certain values of this parameter the model has two vacua. Otherwise, $\mathcal{N}=3$ effective field theory has a single massive vacuum [39]. Below we extend these results to compactifications of M-theory on cotangent bundle of a general two-dimensional Fano surface.[6]

To obtain a three-dimensional field theory with a vacuum moduli space, we can consider adding extra membranes to this configuration. By analogy with the other models, we shall mainly think of a type IIA dual, which is much easier to analyze. In type IIA theory a membrane become a D2-brane. It preserves the same amount of supersymmetry as the original bulk theory, so that the effective theory on the D2-brane is an $\mathcal{N}=3$ abelian gauge theory in 2+1 dimensions. What is this theory? There is an obvious $\mathcal{N}=3$ candidate with $X = T^*\mathbb{CP}^2$ as the Higgs branch [39]:

[6] A Fano variety is a projective variety whose anticanonical class is ample. In complex dimension two, a Fano variety is also called a del Pezzo surface.

Theory: $\mathcal{N} = 4$ vector multiplet with Chern-Simons coupling and 3 hypermultiplets

The sole role of the Chern-Simons coupling is to break supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 3$. Since all hypermultiplets have charge $+1$, the Higgs branch of this theory is described by the following constraints:

$$\sum_{i=1}^{3} |q_i|^2 - |\tilde{q}_i|^2 = m_3 \,, \qquad \sum_{i=1}^{3} q_i\tilde{q}_i = m_1 + im_2 \,. \tag{5.2}$$

From the perspective of the gauge theory, $\vec{m}$ is a FI parameter. The Higgs branch, as written in this form, has a manifest $\mathrm{SU}(3)_F \times \mathrm{U}(1)_R$ symmetry. The simplest way to see that this space is indeed $T^\star\mathbb{CP}^2$ is to set, without loss of generality, $m_1 + im_2 = 0$, so that the variables

$$y_i = \frac{q_i}{\sqrt{|\tilde{q}_i|^2 + m}} \tag{5.3}$$

takes values in a 5-sphere, $\mathbf{S}^5$. Its quotient by the $\mathrm{U}(1)$ gauge action gives us $\mathbb{CP}^2$. The fields $\tilde{q}_i$, constrained by the second equation in (5.2), provide the fibre to give us $X = T^*\mathbb{CP}^2$. The metric on the Higgs branch may be computed explicitly using the hyperkähler quotient construction and is of the toric hyperkähler form,

$$ds^2 = H_{ab}d\vec{\phi}_a \cdot d\vec{\phi}_b + H^{ab}\left(d\sigma_a + \vec{\omega}_{ac}d\vec{\phi}^c\right)\left(d\sigma_b + \vec{\omega}_{bd}\vec{\phi}^d\right), \tag{5.4}$$

where the matrix of harmonic functions is given by,

$$H_{ab} = \frac{1}{|\vec{\phi}_1|}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{|\vec{\phi}_2|}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{|\vec{\phi}_1 + \vec{\phi}_2 + \vec{m}|}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{5.5}$$

These coordinates make manifest only $\mathrm{U}(1)_F^2 \subset \mathrm{SU}(3)_F$, which acts by shifts of σ_a.

While this theory certainly gives the correct moduli space, it needs a little modification to describe correctly a theory on the D2-brane probe, where the dual photon is identified with the M-theory circle. In fact, we expect the theory on the D2-brane probe to realise X as a "mixed" branch: partially Higgs, partially Coulomb. To make this issue more apparent, let us deform X by squashing the T^2 fibers at infinity. This is acheived by gauging the $\mathrm{U}(1)_J$ symmetries [42] as described in section 2. The net result is simply to add to a constant term to the harmonic function,

$$H_{ab} = \begin{pmatrix} e_1^{-2} & 0 \\ 0 & e_2^{-2} \end{pmatrix} + \frac{1}{|\vec{\phi}_1|}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{|\vec{\phi}_2|}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{|\vec{\phi}_1 + \vec{\phi}_2 + \vec{m}|}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{5.6}$$

If we now blow-up the $\mathbb{CP}^2$ zero section to become very large, $\vec{m} \to \infty$, the last term in H decouples and we are left with the product of two Taub-NUT spaces. The theory described above indeed produces this target space in the limit of large FI parameter. However, both copies of this space are in terms of Higgs branch variables. From the discussion above, one of the $\mathbf{S}^1$ fibers is the M-theory circle, we know that one of these copies must arise in a Coulomb branch description. We therefore expect the following two decoupled theories in the $\vec{m} \to \infty$ limit, each of which realizes the Taub-NUT manifold as a different branch of vacuum moduli space

$$\textbf{Coulomb Branch}: \ \mathcal{N}=4 \ \mathrm{U}(1)_1 \text{ with 1 hypermultiplet}$$
$$\textbf{Higgs branch}: \ \mathcal{N}=4 \ \mathrm{U}(1)_2 \times \hat{\mathrm{U}(1)} \text{ with 1 hypermultiplet}$$

In the Higgs branch theory, the hypermultiplet is charged only under $\hat{\mathrm{U}(1)}$, which couples to $\mathrm{U}(1)_2$ via a CS term. To rediscover $T^\star\mathbb{CP}^2$, it is natural to allow the above two theories to interact in a manner which decouples in the limit $\vec{m} \to \infty$. There is a very natural candidate deformation: we couple a further hypermultiplet to $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$, with bare mass $\vec{m}$. Moreover, to break supersymmetry to $\mathcal{N}=3$, we add a CS coupling for $\hat{\mathrm{U}(1)}$

$$\textbf{Theory}: \ \mathrm{U}(1)_1 \times \mathrm{U}(1)_2 \times \hat{\mathrm{U}(1)} \text{ with 3 hypermultiplets}$$

The three hypermultiplets have charges $(1,0,0)$, $(0,0,1)$ and $(1,1,0)$ respectively under the gauge factors, and the third hypermultiplet is assigned a bare mass $\vec{m}$. Moreover, we include a Chern-Simons couplings

$$\hat{A} \wedge (F_2 + \hat{F})\,. \tag{5.7}$$

The latter term breaks supersymmetry to $\mathcal{N}=3$. Note that further self-CS couplings would lift the moduli space of interest and so are disallowed. The scalar potential of this theory is,

$$\begin{aligned} V \,=\, & |\vec{\phi}_1|^2 w_1^\dagger w_1 + |\vec{\hat{\phi}}|^2 w_2^\dagger w_2 + \left|\vec{\phi}_1 + \vec{\phi}_2 + \vec{m}\right|^2 w_3^\dagger w_3 + \\ & + e_1^2 \left|w_1^\dagger \vec{\tau} w_1 + w_3^\dagger \vec{\tau} w_3\right|^2 + e_2^2 \left|w_3^\dagger \vec{\tau} w_3 + \vec{\hat{\phi}}\right|^2 + \hat{e}^2 \left|w_2^\dagger \vec{\tau} w_2 - \vec{\phi}_2\right|^2 . \end{aligned}$$

The vacuum moduli space $V=0$ is given by $\hat{\phi} = w_1 = w_3 = 0$, while ϕ_1, ϕ_2 and w_2 satisfy only the constraint $w_2^\dagger \vec{\tau} w_2 = \vec{\phi}_2$. This branch of vacua is indeed a mixed branch as we anticipated. Classically, the metric is of the form (5.4) with,

$$H_{ab}^{(0)} = \begin{pmatrix} e_1^{-2} & 0 \\ 0 & e_2^{-2} \end{pmatrix} + \frac{1}{|\vec{\phi}_2|} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.8}$$

However, this receives quantum corrections upon integrating out the massive hypermultiplets w_1 and w_3. The final metric is given by the harmonic function (5.6), which indeed reduces to the hyperKähler metric on $T^*\mathbb{CP}^2$ in the limit $e_1^2, e_2^2 \to \infty$.

What of mirror symmetry in our model? There is a version of the "M-theory" flip in the present situation which involves exchanging the two dual photons σ_1 and σ_2. This Z_2 symmetry, far from obvious from the classical lagrangian, is expected to hold at the quantum level whenever $e_1^2 = e_2^2$. Moreover, in the $e_a^2 \to \infty$ limit, it combines with the two $\mathrm{U}(1)_J$ symmetries to yield the full SU(3) isometry of the target space.

Generalizations and IIB models. One may consider more general hyperKähler manifolds of the form:

$$X = T^* B\,,$$

where B is a smooth, compact Fano surface, *i.e.* $B = \mathbb{CP}^1 \times \mathbb{CP}^1$ or a del Pezzo surface $\mathcal{B}_n$. We will be particularly interested in models which admit a toric description of the form (5.4):

$$H_{ab} = \begin{pmatrix} e_1^{-2} & 0 \\ 0 & e_2^{-2} \end{pmatrix} + \frac{1}{|\vec{\phi}_2|}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{i=1}^{n+2} \frac{1}{|p_i \vec{\phi}_1 + q_i \vec{\phi}_2 + \vec{m}_i|}\begin{pmatrix} p_i^2 & p_i q_i \\ p_i q_i & q_i^2 \end{pmatrix}. \tag{5.9}$$

The topology of the hyperKähler toric manifold X is encoded in the toric data; that is, in the pairs of integers (p, q). The space X is a $\mathbf{T}^2$ fibration over $\mathbb{R}^6$, parametrised by two 3-vectors: $\vec{\phi}_1$ and $\vec{\phi}_2$. A 1-cycle in T^2 degenerates at loci in which $\det(H^{-1}) = 0$ or, alternatively, on the loci in which H diverges. The class of the cycle is determined by the values of (p, q). This story is very similar to toric geometry in complex dimension 2, except that the base is 6-dimensional, rather than 2-dimensional. Note, however, that there are only two integers (p, q) which specify 3-planes in the base. In this sense, we lose no information if we think of $\vec{\phi}_{1,2}$ as real numbers, rather than vectors. Then we get a usual toric diagram for the 4-manifold B, which is a $\mathbf{T}^2$ fibration over $\mathbb{R}^2$ parametrised by ϕ_1 and ϕ_2.

Recall that a del Pezzo surface $\mathcal{B}_n$ can be constructed by blowing up n points on $\mathbb{CP}^2$, where $0 \le n \le 8$. If we denote ℓ the class of a line in the original complex projective space $\mathbb{CP}^2$, and E_i $(i = 1, \dots, n)$ the exceptional divisors of the blown up points, then the canonical class of $\mathcal{B}_n$ is given by $K_{\mathcal{B}_n} = -3\ell + \sum_i E_i$. For the first Chern class of $\mathcal{B}_n$ we have:

$$c_1 = -K_{\mathcal{B}_n} = 3\ell - \sum_i E_i\,. \tag{5.10}$$

By definition, c_1 is ample, i.e. it has positive intersection with every effective curve in $\mathcal{B}_n$. It is also useful to know that $\mathcal{B}_n$ for $n \ge 1$ has a description in terms of a fiber space over $\mathbb{CP}^1$. The intersection numbers of the basis elements $\{\ell, E_1, \dots, E_n\} \in H_2(B)$ are:

$$\ell \cdot \ell = 1\,, \qquad E_i \cdot E_j = -\delta_{ij}\,, \qquad \ell \cdot E_i = 0\,. \tag{5.11}$$

Since non-zero Betti numbers of $B = \mathcal{B}_n$ look like:

$$h^0(B) = h^4(B) = 1\,, \qquad h^2(B) = n + 1 \tag{5.12}$$

the models based on cotangent bundles of del Pezzo surfaces are labeled by a number n, which essentially determines the topology of B. Since X is contractible to $B = \mathcal{B}_n$, one has $H^4(X; \mathbb{R}) = H^4(\mathcal{B}_n; \mathbb{R})$, and the non-zero Betti numbers are given by (5.12):

$$h^0(X) = h^4(X) = 1\,, \qquad h^2(X) = n + 1\,. \tag{5.13}$$

As in the case of $T^*\mathbb{CP}^2$ model [39], dynamics of M-theory on these hyperKähler manifolds has many interesting aspects. For example, some of the models are inconsistent unless we introduce background G-fields. Indeed, due to a membrane anomaly, the flux quantization condition requires the period of G-field to be congruent to $c_2(X)$ modulo integers. On the other hand, it is easy to show that $c_2(X)$ for manifolds X of the form T^*B is determined by the topology of B:

$$\int_B \frac{c_2(X)}{2} \cong \int_B c_1^2(B) \mod \mathbb{Z}\,.$$

For our examples, the right-hand side can be easily evaluated using (5.10) and (5.11). The result is:

$$\int_B \frac{G}{2\pi} = k_0 + \frac{9-n}{2}\,, \quad k_0 \in \mathbb{Z}$$

and, in particular, G-flux can not be zero for $B = \mathcal{B}_n$ with n even. The shift in the G-flux quantization condition is expected to be related to the shift in the Chern-Simons coefficient in the effective three-dimensional theory [4]. One would expect a violation of parity symmetry in such theories.

Models with different values of the G-flux can be connected by domain walls, obtained from five-branes wrapped on the 4-cycle B. Hence, they are classified by $H_4(X,\mathbb{Z}) \cong H^4_{\rm cpct}(X,\mathbb{Z})$. Since possible values of the G-flux are classified by $H^4(X,\mathbb{Z})$, the number of models which cannot be connected by domain walls is given by the quotient of these two groups, $H^4(X,\mathbb{Z})/H^4_{\rm cpct}(X,\mathbb{Z})$. By Poincaré duality, the cohomology with compact support is generated by the class $[B]$. On the other hand, $B\cdot B = \chi(B) = n+3$, so that $H^4(X,\mathbb{Z})$ is generated by $[B]/(n+3)$. Hence,

$$H^4(X,\mathbb{Z})/H^4_{\rm cpct}(X,\mathbb{Z}) = \mathbb{Z}_{n+3}\,.$$

Summarizing, different $\mathcal{N}=3$ theories obtained from compactification on $X = T^*B$ are labeled by the value of "flux at infinity" [39]:

$$\Phi_\infty = N + \frac{1}{2}\int_X \frac{G\wedge G}{(2\pi)^2} = N + \frac{1}{2(n+3)}\Big(k_0 + \frac{9-n}{2}\Big)^2, \tag{5.14}$$

where N is the number of space-filling membranes. Given the value of k_0 in a mod $(n+3)$ coset, a vacuum is then found by choosing a non-negative N and an integer k_0 in the given mod $(n+3)$ coset, such that the anomaly condition (5.14) is satisfied. For all del Pezzo surfaces $\mathcal{B}_n$, there is one special case of a model having more than one vacuum (branch of vacua). They appear for $\Phi_\infty = N + (n+3)/8$, with:

$$k_0 = \frac{(n-9)\pm(n+3)}{2}\,.$$

These results generalise [39] to a more general class of hyperKähler manifolds of the form T^*B, where B is a smooth, compact Fano surface. Similarly, it is easy to generalize our $\mathcal{N}=3$ D2-brane probe theory for $T^*\mathbb{CP}^2$ model to reproduce a more general metric like (5.9) on the moduli space:

Theory: $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2 \times \hat{\mathrm{U}(1)}$ with $(n+3)$ hypermultiplets

The first hypermultiplet is charged only under $\hat{U(1)}$, while the remaining $(n+2)$ have charges $(p_i, q_i, 0)$ under $U(1)_1 \times U(1)_2 \times \hat{U(1)}$. As before the theory has CS terms $\hat{A} \wedge (F_2 + \hat{F})$, the latter of which breaks supersymmetry to the desired $\mathcal{N}=3$. It is simple to check that the (mixed) vacuum moduli space of this theory indeed reproduces the metric with harmonic function (5.9).

For a related discussion of M-theory on these (and more general) toric hyperKähler manifolds see [64], where it was shown that these spaces are T-dual to intersecting five-brane configurations in IIB theory which preserve 3/16 fraction of supersymmetry. The constant term in H encodes the asymptotic IIB coupling constant,

$$\frac{1}{g_s} = \frac{e_1 e_2}{e_1^2}\,. \tag{5.15}$$

The second term in (5.9) describes TN space in IIA theory, and is T-dual to an NS5-brane in 12345 directions. The remaining terms are dual to (p_i, q_i)5-branes. We thus have the configuration of intersecting branes,

$$\begin{array}{ll} NS5 & 12345 \\ (p,q)5 & 12[37]_\theta[48]_\theta[59]_\theta\,, \qquad \tan\theta = p/q \end{array}$$

6. Spin(7) holonomy

The most subtle, and also most interesting class of models corresponds to compactification of M-theory on manifold X of Spin(7) holonomy. Such theories use the rich dynamical structure of $\mathcal{N}=1$ three-dimensional gauge theories to the fullest. In this section we shall examine some of only the simplest models, arising from mututally orthogonal intersecting D6-branes. D-brane probes of these models have a world-volume theory in the same class as those described in section 2 and, in particular, do not have any Chern-Simons couplings. We shall describe the moduli space of these compactifications and show that there are the geometrical transitions between branches. However, as in the case of the orthogonal G_2 branes, each of these geometrical transitions is inherited from those of the Calabi-Yau manifold discussed in section 3. We shall further use our mirror symmetry result to conjecture an algebraic quotient construction of these Spin(7) manifolds.

As we pointed out in the previous section, IIA D-brane picture dual to M-theory on eight-dimensional manifolds is typically more involved since the D6-brane locus L often has a conical singularity. There is, however, a simple analog of the orthogonal D6-brane models discussed in sections 4.4. To see this, we calibrate the locus of D6-branes with the following coassociative 4-form,

$$\Psi^{(4)} = *\Psi^{(3)} = e^{4679} + e^{4589} + e^{5678} + e^{3478} + e^{3698} + e^{3579} + e^{3456} \tag{6.1}$$

which restricts to the volume form on any of the seven 4-planes, corresponding to different terms in (6.1). These four-planes have the property that any pair of them intersect over a two-plane. If we pick two D6-branes with corresponding world-volumes (also filling the ubiquitous $x^1 - x^2$ direction), then we reproduce the conifold model of section 3.

For three D6-branes, there are two possibilities: the four-planes may intersect either over a line, or alternatively over a point. In the former case, we return to the G_2 model discussed in the section 4.4. In the latter case, where three four-planes intersect over a point, the configuration breaks $(1/2)^3 = 1/8$ of supersymmetry. The lift to M-theory therefore results in a Calabi-Yau four-fold. We will not discuss this possibility further here.

Therefore the simplest Spin(7) case consists of four D6-branes with the following world-volume directions:

$$\begin{array}{ll} D6_1 & 123456 \\ D6_2 & 123689 \\ D6_3 & 124679 \\ D6_4 & 123579 \end{array}$$

Since all four D6-branes intersect only at a point (the origin, in our notation) in $\mathbb{R}^7$, which is parameterized by x^3, x^4, x^5, x^6, x^7, x^8, x^9, the lift of this configuration to M-theory gives a Spin(7) manifold X with isolated conical singularity. Moreover, since the D6-brane locus L is a collection of four coassociative 4-planes in the flat $\mathbb{R}^7$, it is natural to expect that L can be continuously deformed into a smooth coassociative 4-manifold $L \subset \mathbb{R}^7$. In fact, it is simple to see that we may once again make use of the two deformations of the conifold to smoothen the singularity. Firstly let us consider translating the D6-branes. There are five such parameters,

$$\begin{array}{lll} D6_1 : x^7 = m_1\,, & x^8 = 0\,, & x^9 = 0\,, \\ D6_2 : x^4 = 0\,, & x^5 = 0\,, & x^7 = 0\,, \\ D6_3 : x^3 = 0\,, & x^5 = m_2\,, & x^8 = m_3\,, \\ D6_4 : x^4 = n_1\,, & x^6 = 0\,, & x^8 = n_2\,. \end{array}$$

If $m_i \neq 0$ for $i = 1, 2, 3$ and $n_i \neq 0$ for $i = 1, 2$, then the D6-branes do not intersect and locus L has topology,

$$L = \mathbb{R}^4 \cup \mathbb{R}^4 \cup \mathbb{R}^4 \cup \mathbb{R}^4\,. \tag{6.2}$$

So, from (1.4), we see that the lift to M-theory results in a smooth manifold X of Spin(7) holonomy with three 2-cycles:

$$h_2(X) = 3\,,$$

and all other Betti numbers vanishing.

Exactly as in the G_2 case, we shall see that there are geometric transitions, which involve replacing these 2-cycles with with $\mathbf{S}^3$'s, resulting once again in three topologically distinct manifolds. However, unlike the G_2 case we shall see that we can only blow up a single $\mathbf{S}^3$ if the manifold is to remain non-singular. The analysis is identical to the G_2 case, the only difference being the two pairs of complex structures. If we single out the $D6_1$ and $D6_2$ pair, as well as the $D6_3$ and $D6_4$ pair, it is useful to define the complex combinations,

$$\begin{array}{ll} \psi_1 = x^8 + ix^9\,, & \psi_2 = x^4 + ix^5\,, \\ \tilde{\psi}_1 = x^3 + ix^5\,, & \tilde{\psi}_2 = x^4 + ix^6\,. \end{array}$$

Betti Numbers	Number of Phases	Number of Deformations
$h_2 = 3$, $h_3 = 0$	1	5
$h_2 = 2$, $h_3 = 1$	6	4
$h_2 = 2$, $h_3 = 1$	6	6
...	...	...

Table 4: Some topological phases of a manifold X with Spin(7) holonomy.

We may set $m_1 = 0$, to allow us to deform the first pair on the curve $\psi_1\psi_2 = \rho$. This curve fails to intersect with the $D6_3$ brane if either[7] $m_2 = m_3 = 0$ or, alternatively, if we move the $D6_3$ far enough,

$$m_2^2 m_3^2 - m_2 m_3 \rho_2 - \frac{1}{4}\rho_1^2 > 0\,. \tag{6.3}$$

The same analysis holds for $D6_4$: it fails to intersect the complex curve if either $n_1 = n_2 = 0$ (and $\rho_2 \neq 0$) or

$$n_1^2 n_2^2 - n_1 n_2 \rho_1 - \frac{1}{4}\rho_2^2 > 0\,. \tag{6.4}$$

If we also want no intersection between $D6_3$ and $D6_4$, this requires us to turn on at least, say, n_2. We must therefore move at least one of the branes away from the origin. Of course, one may pick any pair of D6-branes and perform a similar deformations, leading to six branches in each of which the locus L has topology,

$$L = S^1 \times \mathbb{R}^3 \cup \mathbb{R}^4 \cup \mathbb{R}^4 \tag{6.5}$$

which ensures that the lift to M-theory results in a non-compact manifold X with non-trivial Betti numbers:

$$h_2(X) = 2\,, \qquad h_3(X) = 1\,.$$

Finally, as in the G_2 case, we may attempt to resolve the two pairs simultaneously:

$$\psi_1\psi_2 = \rho\,, \qquad \tilde{\psi}_1\tilde{\psi}_2 = \tilde{\rho}\,. \tag{6.6}$$

However, in this case there are no choice of the parameters for which these two complex curves fail to intersect: this therefore always results only in a partial resolution of the Spin(7) singularity.

We catalog different topological phases we found in table 4. However, there might be other phases of our Spin(7) manifold X, and it would be interesting to complete this quest.

Probe theory and the mirror theory. As in the previous sections, it is straightforward to describe the theory on a D2-brane probe of this D6-brane, oriented in the $x^1 - x^2$ plane. We have,

Theory A: U(1) with 7 scalars and 4 hypermultiplets

[7] In the first case, if $\rho_1 \neq 0$, the intersection occurs only asymptotically, as in section 4.4.

The charges of hypermultiplets are all equal to $+1$. If we denote the scalar fields as $\phi_i = x^{i+2}$, the coupling to the hypermultiplets is described by the real superpotential,

$$f = \sum_{i=1}^{4} \left(\vec{A}_i - \vec{M}_i \right) \cdot W_i^\dagger \vec{\tau} W_i \,, < \tag{6.7}$$

where the triplets $\vec{A}_i$ and $\vec{M}_i$ are suitable combinations of the Φ_i and deformation parameters respectively,

$$\vec{A}_1 = (\Phi_9, \Phi_8, \Phi_7)\,, \quad \vec{A}_2 = (\Phi_5, \Phi_4, \Phi_7)\,, \quad \vec{A}_3 = (\Phi_5, \Phi_8, \Phi_3)\,, \quad \vec{A}_4 = (\Phi_6, \Phi_8, \Phi_4) \tag{6.8}$$

and

$$\vec{M}_1 = (0, 0, m_1)\,, \quad \vec{M}_2 = (0, 0, 0)\,, \quad \vec{M}_3 = (m_2, m_3, 0)\,, \quad \vec{M}_4 = (0, n_2, n_1)\,. \tag{6.9}$$

As in previous sections, we have included only the translational deformations corresponding to blowing up $\mathbf{S}^2$'s. Further deformations, corresponding to blowing up $\mathbf{S}^3$'s are given by the usual terms (3.9). At the singular point, $m_i = n_i = 0$, the theory has three global $\mathrm{U}(1)_C$ flavor symmetries coming from gauge symmetries on the D6-branes (not including the diagonal one). As usual, there is also a $\mathrm{U}(1)_J$ symmetry corresponding to shift of the dual photon. Therefore, upon lift to M-theory we expect a Spin(7) conical singularity with the global symmetry group $K_Y = \mathrm{U}(1)_J \times \mathrm{U}(1)_C^3$. Since the flavor symmetries are unbroken by the mass parameters m_i, n_i, this is expected to survive a deformation to a smooth manifold X, with all singular points replaced by $\mathbf{S}^2$'s. The same is not true on the other branches where we blow up some $\mathbf{S}^3$'s.

Recall that in the previous sections, we were able to derive an algebraic expression for the manifold X by dualising to a configuration of NS5-branes in IIB. In the present case, this is only possible if we T-dualise along x^1 or x^2 directions. In this case, the D2-brane turns into a D-string as opposed to a D3-brane, and we are unable to read off the mirror theory. Nevertheless, it is still possible to derive the mirror theory using the field theory techniques of section 2.1. As we mentioned above, these reproduce the same Higgs branch as the string picture for the G_2 case. Here we must rely on these techniques for want of another method. We find,

Theory B: $\mathrm{U}(1)^3$ with 5 scalars and 4 hypermultiplets

The charges of the hypermultiplets are $(+1, -1, 0, 0)$, $(0, +1, -1, 0)$, and $(0, 0, +1, -1)$. The theory consists of only Yukawa couplings, and no quartic (or higher) superpotentials. These Yukawa couplings are given by,

$$\begin{aligned} f_{\mathrm{Yuk}} = {} & \Phi_1 \left(W_1^\dagger \tau^3 W_1 - W_2^\dagger \tau^3 W_2 - m_1 \right) + \Phi_2 \left(W_2^\dagger \tau^1 W_2 - W_3^\dagger \tau^1 W_3 - m_2 \right) + \\ & + \Phi_3 \left(W_3^\dagger \tau^2 W_3 - W_1^\dagger \tau^2 W_1 - m_3 \right) + \Phi_4 \left(W_2^\dagger \tau^2 W_2 - W_4^\dagger \tau^3 W_4 - n_1 \right) + \\ & + \Phi_5 \left(W_1^\dagger \tau^2 W_1 - W_4^\dagger \tau^2 W_4 - n_2 \right). \end{aligned} \tag{6.10}$$

Note that the first three terms coincide with the G_2 Yukawa couplings (2.22). The last two terms are unique to this Spin(7) example. The Higgs branch of Theory B is parameterized by the 16 real variables in the hypermultiplets, subject to 3 U(1) gauge orbits and the 5 constraints,

$$\begin{aligned} |q_1|^2 - |\tilde{q}_1|^2 - |q_2|^2 + |\tilde{q}_2|^2 &= m_1 \\ \mathrm{Re}(\tilde{q}_2 q_2 - \tilde{q}_3 q_3) &= m_2 \\ \mathrm{Im}(\tilde{q}_3 q_3 - \tilde{q}_1 q_1) &= m_3 \\ \mathrm{Im}(\tilde{q}_1 q_1 - \tilde{q}_4 q_4) &= n_2 \\ \mathrm{Im}(\tilde{q}_2 q_2) - |q_4|^2 + |\tilde{q}_4|^2 &= n_1 \end{aligned}$$

Note that, unlike the G_2 cases, the final equation relates different components of the 3-vectors $W^\dagger\vec{\tau}W$.

Further D6-branes. As is clear from the coassociative form (6.1), there exist further loci L consisting of $N = 5, 6$ or 7 D6-branes, each of which lifts to M-theory on a Spin(7) manifold. Here we describe briefly only the case of $N = 7$, with other cases (including the $N = 4$ model described above) arising as limiting cases. The D6-branes have world-volumes,

$$\begin{array}{ll} D6_1 & 123456 \\ D6_2 & 123689 \\ D6_3 & 124679 \\ D6_4 & 123579 \\ D6_5 & 125678 \\ D6_6 & 124589 \\ D6_7 & 123478 \end{array}$$

It is a simple matter to determine the possible resolutions of this singularity by implementing the resolution and deformation of the conifold pairwise. For example, there are 14 parameters arising from moving the D6-branes away from the origin. If all D6-branes remain flat, but non-intersecting, then the locus L lifts to a Spin(7) manifold X with homology $h_2(X) = 6$, and no further cycles. As in the previous case of four D6-branes, we may blow down any one of these $\mathbf{S}^2$'s and replace it with a $\mathbf{S}^3$, by picking any pair of D6-branes and deforming them on a complex curve. If the other D6-branes are moved far enough from the origin, the M-theory lift is once again smooth. After passing through this geometrical transition, X has Betti numbers:

$$h_2(X) = 5\,, \qquad h_3(X) = 1\,.$$

Finally, we saw in the case of four D6-branes that it was not possible to deform two pairs of D6-branes simultaneously without the complex curves intersecting. The same is true here.

It is straightforward to write down the theory on a probe D2-brane

Theory A: U(1) with 7 scalars and 7 hypermultiplets with the Yukawa couplings determined, as usual, by the orientation of the D6-branes. Once again, we are forced to use field theoretic techniques to determine the mirror theory,

Theory B: $\mathrm{U}(1)^6$ with 14 scalars and 7 hypermultiplets

where the hypermultiplets have the usual alternating $+1, -1$ charges under the gauge group, and the superpotential consists only of Yukawa couplings, which result in 14 real constraints on the 14 complex degrees of freedom in the hypermultiplets. These constraints may be written in compact form,

$$\mathrm{Re}(\tilde{q}_i q_i) = \mathrm{Im}(\tilde{q}_{i+4} q_{i+4}) = |q_{i+6}|^2 - |\tilde{q}_{i+6}|^2 \qquad i = 1, \ldots, 7 \tag{6.11}$$

where i is defined modulo 7; i.e. $q_{i+7} \equiv q_i$. The 14 translational parameters equate each of these terms up to a constant.

As in previous cases, we may easily generalize this configuration by having N_i of each branes, so that when the resulting Spin(7) manifold X develops a conical singularity, there is an enhanced $\prod_i \mathrm{U}(N_i)$ gauge symmetry.

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A. $\mathcal{N} = 1$ Chern-Simons mirrors

In section 2 of this paper we derived a large class of abelian mirror pairs with $\mathcal{N} = 1$ supersymmetry. However, in each case all charged matter fields lie within hypermultiplets, ensuring invariance under charge conjugation. From the field theory perspective, this restriction arose because we derived the mirror pairs from deforming $\mathcal{N} = 4$ flavor symmetries which act symmetrically upon the two chiral multiplets Q and $\tilde{Q}$.

In this appendix, we show how to relax this constraint and obtain $\mathcal{N} = 1$ three-dimensional mirror pairs with only chiral (as opposed to hyper-) multiplets. However, we have not yet found a place for these theories in our probe set-ups, and they play no role in the bulk of this paper. We suspect they may be important in more complicated Spin(7) compactifications, and include them here only for completeness.

The prescription we use was given in [45], where the $\mathcal{N} = 4$ mirror pairs were deformed by gauging the R-symmetry currents. Specifically, an abelian R-symmetry contained within the diagonal group of $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_N$ may be gauged preserving $\mathcal{N} = 2$ supersymmetry. This gives a mass splitting not only to the $\mathcal{N} = 4$ vector multiplets, as in the previous subsection, but also to the $\mathcal{N} = 4$ hypermultiplets. As a result, certain charged chiral multiplets become heavy and decouple, leaving behind a reminder of their presence in the form of CS couplings $\kappa^{ab} A^a \wedge F^b$. The resulting $\mathcal{N} = 2$ mirror pairs had been previously discovered in [48],

Theory A : $\mathrm{U}(1)^r$ with N chiral multiplets

Theory B : $\hat{\mathrm{U}(1)}^{N-r}$ with N chiral multiplets

As in the section 2, the chiral multiplets of the two theories carry charges R and $\hat{R}$ respectively, satisfying

$$\sum_{i=1}^{N} R_i^a \hat{R}_i^p = 0\,. \tag{A.1}$$

The new element in these theories is the presence of CS couplings, κ and $\hat{\kappa}$ for the Theories A and B respectively,

$$\kappa^{ab} = \frac{1}{2}\sum_{i=1}^{N} R_i^a R_i^b\,; \qquad \hat{\kappa}^{ab} = \frac{1}{2}\sum_{i=1}^{N} \hat{R}_i^a \hat{R}_i^b\,. \tag{A.2}$$

Including all possible mass and FI parameters, the scalar potential for Theory A takes the form,

$$V_A = \sum_{a=1}^{r} e_a^2 \left(R_i^a |q_i|^2 - \frac{1}{2} R_i^a R_i^b \phi^b - \zeta^a - \kappa_{ab}\phi^b \right)^2 + \sum_{i=1}^{N} (R_i^a \phi^a + m_i)^2 |q_i|^2 \tag{A.3}$$

with a similar tale for Theory B, with parameters $\hat{\zeta}$ and $\hat{m}$. These are determined by the mirror map [47],

$$\zeta^a - \frac{1}{2} R_i^a m_i = R_i^a \hat{m}_i\,; \qquad \hat{\zeta}^p + \frac{1}{2}\hat{R}_i^p \hat{m}_i = -\hat{R}_i^p m_i\,. \tag{A.4}$$

The difference between this and the $\mathcal{N} = 4$ mirror map (2.4) may be traced to a finite renormalization of the FI parameters. As in previous cases, the two theories as stated are mirror only in the strong coupling limit $e_a^2 \to \infty$. However, there exists a deformation to "Theory B′" which is conjectured to be valid at all energy scales [45], thus justifying further attempts to deform these theories. In fact, these deformations proceed as in section 2, resulting in $\mathcal{N} = 1$ Maxwell-Chern-Simons mirror pairs,

Theory A : $\mathrm{U}(1)^r$ with k scalars and N chiral multiplets

Theory B : $\hat{\mathrm{U}(1)}^{N-r}$ with $N - k$ scalars and N chiral multiplets

The charges and CS couplings are as above. We further have a real superpotential leading to a Yukawa coupling and real masses for Theory A,

$$f = S_i^\alpha \Phi_\alpha Q_i^\dagger Q_i + M^{\alpha\beta}\Phi_\alpha \Phi_\beta \tag{A.5}$$

with coupling constants given by the maximal rank matrix S. The Yukawa couplings and real masses for Theory B are denoted by $\hat{S}$ and $\hat{M}$ respectively. The former satisfy equation (2.10) from section 2,

$$\sum_{i=1}^{N} S_i^\alpha S_i^\rho = 0 \tag{A.6}$$

while the real masses for both Theory A and B are determined in terms of the Yukawa couplings,

$$M^{\alpha\beta} = \frac{1}{2} S_i^\alpha S_i^\beta \, ; \qquad \hat{M}^{\rho\lambda} = \frac{1}{2} \hat{S}_i^\rho \hat{S}_i^\lambda \, . \tag{A.7}$$

Since the both FI parameters and mass parameters are associated with scalar, rather than vector, multiplets in $\mathcal{N}=1$ theories, the renormalization of ζ depends on S rather than R, and the mirror map (A.4) becomes,

$$\zeta^\alpha - \frac{1}{2} S_i^\alpha m_i = R_i^\alpha \hat{m}_i \, ; \qquad \hat{\zeta}^\rho + \frac{1}{2} \hat{R}_i^\rho \hat{m}_i = -\hat{R}_i^\rho m_i \, . \tag{A.8}$$

Let us focus on the Coulomb branch of Theory A, parameterized by k real scalars and r dual photons. It naturally described by a k real dimensional base space Q, fibered with the torus $\mathbf{T}^r$. It exists only if the CS couplings κ, the real masses M and the FI parameters ζ all vanish. However, unlike the situation with hypermultiplets, each of these quantities receives a correction at one-loop, and we require this quantum corrected parameter to vanish. The shifts are,

$$\begin{aligned} \kappa^{ab} &\to \kappa^{ab} - \frac{1}{2} \sum_i R_i^a R_i^b \, \mathrm{sign}\, M_i \\ M^{\alpha\beta} &\to M^{\alpha\beta} - \frac{1}{2} \sum_i S_i^\alpha S_i^\beta \, \mathrm{sign}\, M_i \\ \zeta^\alpha &\to \zeta^a - \frac{1}{2} \sum_i S_i^\alpha m_i \, \mathrm{sign}\, M_i \end{aligned}$$

where

$$M_i = S_i^\alpha \phi_\alpha + m_i \tag{A.9}$$

is the effective mass of the i^{th} chiral multiplet. We see that the Coulomb branch exists for $\zeta = \frac{1}{2} S_i^\alpha m_i$ and for ϕ_α restricted to lie within the range $M_i \geq 0$. This describes the base Q. At the boundaries of Q, given by $M_i = 0$, a cycle of $\mathbf{T}^r$, corresponding to the linear combination $R_i^a \sigma_a$, degenerates. In contrast, the Higgs branch of Theory B is given by the vanishing of the D-term, $\hat{S}_i^\rho |q_i|^2 = \hat{\zeta}^\rho = \hat{S}_i^\rho m_i$, modulo the gauge action, $q_i \to \exp(\hat{R}_i^p c_p) q_i$. Comparing to the Coulomb branch, we have the mirror map between fields,

$$|q_i|^2 = M_i \, ; \qquad 2\mathrm{arg}\ (q_i) = R_i^a \sigma_a \, . \tag{A.10}$$

As explained in previous sections, any attempt to derive mirror symmetry for theories with only $\mathcal{N}=1$ supersymmetry is necessarily conjectural, since there are few quantitative tests available. Nevertheless, here we present a simple example which illustrates how mirror symmetry works in this case.

An example: $\mathbf{S}^3$. One advantage of using chiral multiplets is that we may engineer the compact Coulomb branches. As an example we examine the case of $\mathbf{S}^3$ in detail. As a Higgs branch of Theory B, this is extremely easy to construct,

Theory B: 1 scalar and 2 chiral multiplets

Each chiral multiplet has Yukawa coupling +1 with the real scalar. A FI parameter 2ζ yields the D-term constraint,

$$|q_1|^2 + |q_2|^2 = 2\zeta \tag{A.11}$$

which is the $\mathbf{S}^3$ of interest. The theory has a global SO(4) symmetry group, which protects the metric from quantum corrections. It will prove useful to change coordinates to $q_i = r_i \exp(i\alpha_i)$. Then defining $\phi = r_1^2 - \zeta = \zeta - r_2^2$, we see that this coordinate is restricted to the interval

$$-\zeta \leq \phi \leq \zeta\,. \tag{A.12}$$

We may now view $\mathbf{S}^3$ as a fibration of $\mathbf{T}^2$ over this interval, with the round metric given by,

$$2ds^2 = \frac{\zeta}{\zeta^2 - \phi^2} d\phi^2 + 2(\zeta + \phi)d\alpha_1^2 + 2(\zeta - \phi)d\alpha_2^2\,. \tag{A.13}$$

Our task is to reconstruct $\mathbf{S}^3$ as the Coulomb branch. The mirror theory is,

Theory A: $\mathrm{U}(1)^2$ with 1 scalar and 2 chiral multiplets.

where each chiral has charge +1 under only a single U(1) factor, and the Yukawa coupling is $S = (+1, -1)$. This theory has vanishing FI coupling, but each chiral is assigned a real mass $m_1 = m_2 = \zeta$. Moreover, there is a real mass $M = -1$ for ϕ and a CS coupling $\kappa^{ab} = -\frac{1}{2}\delta^{ab}$. Following the prescription above, we see that the Coulomb branch exists for $M_i \geq 0$, which translates precisely to the interval (A.12). We can begin to calculate the metric perturbatively. The relevant dimensionless expansion parameters are $e^2/|\zeta \pm m|$. At one-loop we find,

$$ds^2 = \left(\frac{1}{e^2} + \frac{\frac{1}{2}}{\zeta - \phi} + \frac{\frac{1}{2}}{\zeta + \phi}\right) d\phi^2 + \left(\frac{1}{e^2} + \frac{\frac{1}{2}}{\zeta - \phi}\right)^{-1} d\sigma_1^2 + \left(\frac{1}{e^2} + \frac{\frac{1}{2}}{\zeta + \phi}\right)^{-1} d\alpha_2^2\,. \tag{A.14}$$

With $\mathcal{N} = 1$ supersymmetry, one would expect this metric to receive many more quantum corrections, especially in the neighborhood $\phi \to \pm\zeta$, where we have integrated out light matter fields. Nonetheless, it is amusing to note that in the strong coupling limit $e^2 \to \infty$, this metric reproduces the round metric on $\mathbf{S}^3$ (A.13). In this limit the manifest $\mathrm{U}(1)_J^2$ symmetry of the Coulomb branch is enhanced to SO(4). It is tempting to conjecture that this non-abelian infra-red symmetry protects the Coulomb branch from further corrections.

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